Written portion of the Candidacy Examination

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Structure of the document. The main body of the document (Sections 1 to 6) gives a summary of the examinee's research project in ergodic Ramsey theory, which will lead to his PhD thesis directed by Prof. Vitaly Bergelson. Sections 1 to 3 and 5 summarize the student's perspective of some concepts in the area of ergodic Ramsey theory, like Furstenberg's correspondence principle and sets of returns. Sections 4 and 6 contain selected topics from the recent and current research of the examinee, along with questions which can serve as a guide for future progress.

Appendices A and B include proofs of some technical results used in the main body.

Finally, in Section M we briefly summarize the findings of the research project of the student under the guidance of Prof. Facundo Mémoli in the area of metric geometry.

In this document, by *m.p.s.* (probability measure preserving system) we mean a quadruplet $(X, \mathcal{B}, \mu, T)$, where X is a set, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a σ-algebra, μ is a probability measure on (X, μ), $T : X \to X$ is measurable and $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.

1 Ultrafilters and limits

We briefly introduce some basic notions about ultrafilters which we will use in Remark 2.7, Theorem 3.3, Proposition 5.2 and Appendix A.

Definition 1.1. An ultrafilter in a set X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

1. $X \in \mathcal{F}, \varnothing \notin \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
3. For all $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.
4. For all $A \subseteq X$, $A \in \mathcal{F}$ if and only if $X \setminus A \notin \mathcal{F}$.

Remark 1.2. Ultrafilters correspond to finitely additive, $\{0, 1\}$-valued probability measures μ on $\mathcal{P}(\mathbb{N})$ (letting $\mu(A) = 1$ if $A \in \mathcal{F}$ and $\mu(A) = 0$ if not). We say that $\mathcal{F}$-*almost all* natural numbers n satisfy a property P if $\{n \in \mathbb{N}; P(n)\} \in \mathcal{F}$. In particular,

'$\mathcal{F}$-almost all n satisfies P or Q' $\iff$ '$\mathcal{F}$-almost all n satisfies P or $\mathcal{F}$-almost all n satisfies Q'

'$\mathcal{F}$-almost all n satisfies P and Q' $\iff$ '$\mathcal{F}$-almost all n satisfies P and $\mathcal{F}$-almost all n satisfies Q'

We can use ultrafilters to define limits:

Proposition 1.3. *Let X be a set with an ultrafilter $\mathcal{F} \subseteq \mathcal{P}(X)$ and let Y be a compact, Hausdorff space. Then for each function $f : X \to Y$ there exists a unique $y_0 \in Y$ such that, for all neighborhoods U of Y_0, we have $f^{-1}(U) \in \mathcal{F}$. We denote $y_0 = \lim_{x \to \mathcal{F}} f(x)$.*

Proof. We first prove existence of y_0: if for all $y \in Y$ there is a nhood U_y such that $f^{-1}(U_y) \notin \mathcal{F}$, then as Y is compact, we would have $Y = U_{y_1} \cup \cdots \cup U_{y_n}$ for some $y_1, \ldots, y_n \in Y$, so $X = f^{-1}(U_{y_1}) \cup \cdots \cup f^{-1}(U_{y_n}) \notin \mathcal{F}$, a contradiction. Uniqueness of y_0 follows from the fact that Y is Hausdorff, so if there were two limits y_0, y_0', we would have disjoint neighborhoods U, U' such that $f^{-1}(U), f^{-1}(U') \in \mathcal{F}$, so $\varnothing = f^{-1}(U) \cap f^{-1}(U') \in \mathcal{F}$, a contradiction. □

In particular, a fixed ultrafilter $\mathcal{F}$ associates to all bounded sequences $(x_n)_{n\in\mathbb{N}}$ of real numbers a limit $\lim_{n\to\mathcal{F}} x_n$, which satisfies usual limit properties such as $\lim_{n\to\mathcal{F}}(ax_n + by_n) = a\lim_{n\to\mathcal{F}} x_n + b\lim_{n\to\mathcal{F}} y_n$, but not others such as $\lim_{n\to\mathcal{F}} x_n = \lim_{n\to\mathcal{F}} x_{n+1}$ (consider the sequence $x_n = (-1)^n$).

Proposition 1.4. *Let $\mathcal{G} \subseteq \mathcal{P}(X)$. Then there exists an ultrafilter $\mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{P}(X)$ if and only if all finite intersections of elements of $\mathcal{G}$ are nonempty.*

Proof. $\Longrightarrow$ is obvious. for $\Longleftarrow$, let $\mathcal{G}_1 \subseteq \mathcal{P}(X)$ be the family of finite intersections of elements of $\mathcal{G}$, and let $\mathcal{G}_2 \subseteq \mathcal{P}(X)$ be the family of all sets containing some element of $\mathcal{G}_1$ (so $\mathcal{G} \subseteq \mathcal{G}_2$). It is easy to check that $\mathcal{G}_2$ is a filter (i.e. it satisfies properties 1,2,3 in Definition 1.1). A standard argument using Zorn's lemma shows that all filters are contained in some ultrafilter, so we are done. □

2 Notions of density

2.1 Densities in $\mathbb{N}$

The concepts we know as 'density' associate to each subset of natural numbers a constant in $[0, 1]$, its density, with the set being 'large' if it has positive density. Let us see some examples of densities.

Definition 2.1. The *upper density* and *lower density* of a set $A \subseteq \mathbb{N}$ are defined respectively as

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1, \ldots, N\}|}{N}, \tag{1}$$

$$\underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap \{1, \ldots, N\}|}{N}. \tag{2}$$

If both quantities coincide, we denote them by $d(A)$, the *natural density* of A.

Fix a non-principal ultrafilter $\mathcal{F}$ in $\mathbb{N}$ (see Section 1), we define as follows the $\mathcal{F}$*-density* of A:

$$d_{\mathcal{F}}(A) = \lim_{N\to\mathcal{F}} \frac{|A \cap \{1, \ldots, N\}|}{N}. \tag{3}$$

Remark 2.2. The densities from Definition 2.1 are all constructed as limits (a different kind of limit for each density) when N goes to infinity of the sequence of probability measures μ_N on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ given by $\mu_N(A) = \frac{|A\cap\{1,\ldots,N\}|}{N}$.

Of course, $d, \overline{d}, \underline{d}$ and $d_{\mathcal{F}}$ take values in the interval $[0, 1]$.

Proposition 2.3. *The density $d_{\mathcal{F}}$ satisfies the following for any $A, B \subseteq \mathbb{N}$:*

1. *If $d(A)$ is defined, then $d_{\mathcal{F}}(A) = d(A)$.*
2. *$\underline{d}(A) \leq d_{\mathcal{F}}(A) \leq \overline{d}(A)$ for all A.*
3. *$d_{\mathcal{F}}(A) = d_{\mathcal{F}}(A + 1)$, where $A + 1 = \{a + 1; a \in A\}$.*

4. *If A, B are disjoint, then $d_{\mathcal{F}}(A \cup B) = d_{\mathcal{F}}(A) + d_{\mathcal{F}}(B)$. That is, $d_{\mathcal{F}}$ is a finitely additive probability measure in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.*

Note that natural density satisfies the same properties as $d_{\mathcal{F}}$, except that it is not defined for all $A \subseteq \mathbb{N}$, and upper density is defined for all $A \subseteq \mathbb{N}$ but is not a finitely additive probability measure in $\mathbb{N}$, although we do have

$$\overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B).$$

Thus, both $\overline{d}$ and $d_{\mathcal{F}}$ are *partition regular*, that is, if $\overline{d}(A) > 0$ and $A = A_1 \cup \cdots \cup A_n$, then $\overline{d}(A_i) > 0$ for some i. The following fact will be used to prove the Furstenberg correspondence principle:

Proposition 2.4. *For all $A \subseteq \mathbb{N}$ there is an ultrafilter $\mathcal{F}$ such that $d_{\mathcal{F}}(A) = \overline{d}(A)$.*

Proof. Let $(N_k)_k$ be a sequence such that $\overline{d}(A) = \lim_{k\to\infty} \frac{|A\cap\{1,\ldots,N_k\}|}{N_k}$.

Consider the family of sets $A_n = \{N_k; k \geq n\}$. Note that all finite intersections of the sets A_n are nonempty, so by Proposition 1.4 there is an ultrafilter $\mathcal{F}$ such that $A_n \in \mathcal{F}$ for all n. We then have $d_{\mathcal{F}}(A) = \overline{d}(A)$, as we wanted. □

2.2 Densities in amenable groups

We recall some basics about countable amenable groups, a class of groups to which notions of density naturally generalize. In this document we will be interested in countable, discrete groups.

Definition 2.5. A left-*Følner sequence* in a countable group G is a sequence $(F_N)_N$ of finite subsets of G such that $\lim_{N\to\infty} \frac{|F_N \Delta g F_N|}{|F_N|} = 0$ for all $g \in G$. A countable group G is *left-amenable* if it has a left-Følner sequence.

From now, we will just say 'Følner sequence' and 'amenable' instead of 'left-Følner sequence' and 'left-amenable'. One can define right analogues using right translates $F_N g$ instead of gF_N.

Examples 2.6. Examples of Følner sequences are $F_N = \{1, \ldots, N\}$ in $\mathbb{Z}$, $F_N = \{1, \ldots, N\} \times \{1, \ldots, N^2\}$ in $\mathbb{Z}^2$, $F_N = A_5$ in the alternating group A_5. See Example 6.20 for an especially nice Følner sequence in $\mathbb{Q}$.

Countable abelian groups are amenable. Finite groups are amenable. Subgroups and quotients of amenable groups are amenable, and if N is a normal subgroup of G and both N and G/N are amenable, then G is amenable. Groups freely generated by more than one element are not amenable.

Remark 2.7. If a countable group G is amenable, then it has a finitely additive left-invariant probability measure[1], that is, a finitely additive probability measure μ on $(G, \mathcal{P}(G))$ such that $\mu(gA) = \mu(A)$ for all $g \in G, A \subseteq G$. Indeed, we construct such a measure $d_{F,\mathcal{F}}$ below.

Definition 2.8. Given a countable amenable group G with a Følner sequence $F = (F_N)_N$, the *upper F-density* and *lower F-density* of $A \subseteq G$ are given by

$$\overline{d}_F(A) = \limsup_{N\to\infty} \frac{|A \cap F_N|}{|F_N|}, \tag{4}$$

$$\underline{d}_F(A) = \liminf_{N\to\infty} \frac{|A \cap F_N|}{|F_N|}. \tag{5}$$

[1] Reciprocally, groups with finitely additive, left-invariant probability measures are amenable. This is how one usually proves that groups freely generated by more than one element are not amenable.

If both quantities coincide, we denote them by $d_F(A)$, the *F-density* of A. For any non-principal ultrafilter $\mathcal{F}$ in $\mathbb{N}$, we denote

$$d_{F,\mathcal{F}}(A) = \lim_{N\to\mathcal{F}} \frac{|A\cap F_N|}{|F_N|}. \tag{6}$$

Note that as F is a Følner sequence, as opposed to just any sequence of finite subsets of G, then all the notions of density from Definition 2.8 are left-invariant: for all $A\subseteq G, g\in G$ we have $\overline{d}_F(A) = \overline{d}_F(gA)$. We do not always have $\overline{d}_F(A) = \overline{d}_F(Ag)$ if F is not a right-Følner sequence.

Definition 2.9 (Banach density)**.** Given a countable amenable group G and $A\subseteq G$, we define the *upper/lower Banach densities* of A as

$$d^*(A) = \sup\left\{\overline{d}_F(A); F \text{ Følner sequence in } G\right\} = \sup\left\{\underline{d}_F(A); F \text{ Følner sequence in } G\right\}.$$
$$d_*(A) = \inf\left\{\underline{d}_F(A); F \text{ Følner sequence in } G\right\} = \inf\left\{\overline{d}_F(A); F \text{ Følner sequence in } G\right\}.$$

If $d^*(A) = d_*(A)$, we will call this value the *uniform density* of A, $d_u(A)$. One can show that for all $A\subseteq\mathbb{Z}$ we have

$$d^*(A) = \limsup_{N-M\to\infty} \frac{|A\cap\{N+1,N+2,\dots,M\}|}{N-M}.$$

Here, by $\lim_{N-M\to\infty} f(N,M) = k$ we mean that for all $\varepsilon>0$ there exists $n\in\mathbb{N}$ such that for all pairs (N,M) such that $N-M>n$ we have $|f(N,M)-k|<\varepsilon$.

3 Furstenberg's correspondence principles

In this section we state and prove several versions of the Furstenberg correspondence principle (FCP from now), using the machinery from Appendix A. The FCP provides a bridge between combinatorics and measure theory. It gives, for each 'big' set of natural numbers with respect to some density, a measure theoretic object with similar properties.

Theorem 3.1 (FCP, see [Be97, Theorem 6.4.17])**.** *Let G be a countable amenable group with a Følner sequence $F=(F_N)_N$, let $A\subseteq G$. Then there is a m.p.s. $(X,\mathcal{B},\mu,(T_g)_{g\in G})$ and $B\in\mathcal{B}$ with $\mu(B) = \overline{d}_F(A)$ such that for all $k\in\mathbb{N}$ and $h_1,\dots,h_k\in G$ one has*

$$\overline{d}_F(h_1A\cap\dots\cap h_kA) \geq \mu(T_{h_1}(B)\cap\dots\cap T_{h_k}(B)) \tag{7}$$

The result still holds if we change both appearances of $\overline{d}_F$ by d^*.

Note that in Equation (7) we have inequalities instead of equalities; this is only necessary[2] because $\overline{d}_F$ and d^* are not finitely additive probability measures. Once we replace $\overline{d}_F, d^*$ by finitely additive probability measures, we get versions of the FCP with equality. Here are two examples (a similar statement can be found in [BM98, Theorem 2.1]):

Theorem 3.2 (FCP for natural density)**.** *Let G be a countable amenable group with a Følner sequence $F=(F_N)_N$ and $A\subseteq G$. Then there exists a m.p.s. $(X,\mathcal{B},\mu,(T_g)_{g\in G})$ and $B\in\mathcal{B}$ such that, for all $k\in\mathbb{N}$ and $h_1,\dots,h_k\in G$, whenever $d_F(h_1A\cap\dots\cap h_kA)$ is defined we have*

$$d_F(h_1A\cap\dots\cap h_kA) = \mu(T_{h_1}(B)\cap\dots\cap T_{h_k}(B)). \tag{8}$$

[2]FCP with equality is false; it is not hard to find a set $A\subseteq\mathbb{N}$ such that $\overline{d}(A) < \overline{d}(A\cap(A+2)) + \overline{d}(A\cap(A+3)) - \overline{d}(A\cap(A+2)\cap(A+3))$, e.g. let $A = \left(2\mathbb{Z}\cap\bigcup_n\left[2^{2^{2n}}, 2^{2^{2n+1}}\right]\right)\cup\left(3\mathbb{Z}\cap\bigcup_n\left[2^{2^{2n+1}}, 2^{2^{2n+2}}\right]\right)$. But of course, any sets B_1,B_2,B_3 in a probability space will satisfy $\mu(B_1)\geq\mu(B_1\cap B_2)+\mu(B_1\cap B_3)-\mu(B_1\cap B_2\cap B_3)$.

Theorem 3.3 (FCP for ultrafilter density)**.** *Let G be a countable amenable group with a Følner sequence $F = (F_N)_N$, $A \subseteq G$ and $\mathcal{F}$ a non-principal ultrafilter in $\mathbb{N}$. Then there exists a m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and $B \in \mathcal{B}$ such that, for all $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in G$,*

$$d_{F,\mathcal{F}}(h_1 A \cap \cdots \cap h_k A) = \mu(T_{h_1}(B) \cap \cdots \cap T_{h_k}(B)). \tag{9}$$

Proof of Theorems 3.1 and 3.2 from Theorem 3.3. To prove Theorem 3.2 just choose any non-principal ultrafilter $\mathcal{F}$ and note that $d_{F,\mathcal{F}}(h_1 A\cap\cdots\cap h_k A) = d_F(h_1 A\cap\cdots\cap h_k A)$ whenever $d(h_1 A\cap\cdots\cap h_k A)$ is defined, so by Theorem 3.3 we are done.

To prove Theorem 3.1 for upper density note that by Proposition 2.4 there is an ultrafilter $\mathcal{F} \in \beta\mathbb{N}$ such that $d_{\mathcal{F}}(A) = \overline{d}(A)$. Now let $(X, \mathcal{B}, \mu, T)$, let B be as in Theorem 3.3, so that for all $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in G$,

$$\overline{d}_F(h_1 A \cap \cdots \cap h_k A) \geq d_{F,\mathcal{F}}(h_1 A \cap \cdots \cap h_k A) = \mu(T_{h_1}(B) \cap \cdots \cap T_{h_k}(B)).$$

The proof for d^* is similar, using that for any $A \subseteq G$ there is a Følner sequence $(F_N)_N$ such that $d^*(A) = d_F(A)$, so there is some ultrafilter $\mathcal{F}$ such that $d_{F,\mathcal{F}}(A) = d^*(A)$. □

Theorem 3.3 is a concrete case of the much more general statement given below. The main takeaway of the statement is that finitely additive and countably additive measures behave in the same way with respect to finite unions, intersections and complements.

Proposition 3.4. [3] *Let X_0 be a set with an algebra $\mathcal{B}_0 \subseteq \mathcal{P}(X_0)$, a finitely additive probability measure $\mu_0 : \mathcal{B}_0 \to [0, 1]$ and a measure preserving action $(S_g)_{g\in G}$ of a group*[4] *G on $(X_0, \mathcal{B}_0, \mu_0)$. Then there is a m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and a set map $\Phi : \mathcal{B}_0 \to \mathcal{B}$ such that for all $A, B \in \mathcal{B}_0$ and $g \in G$,*

1. $\mu(\Phi(B)) = \mu_0(B)$.
2. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$, $\Phi(X_0 \setminus A) = X \setminus \Phi(A)$, $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.
3. $\Phi(S_g(A)) = T_g(\Phi(A))$.

Remark 3.5. Note that we do not necessarily have $\Phi\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} \Phi(A_n)$. Indeed, that property must fail if μ_0 is not countably additive.

Proof. Let $\mathcal{F}$ be a non-principal ultrafilter in $\mathbb{N}$. Let $(X, \mathcal{B}, \mu, (T_g)_{g\in G}) = \lim_{n\to\mathcal{F}}(X_0, \mathcal{B}_0, \mu_0, (S_g)_{g\in G})$, as defined in Definition A.7, and let $\Phi : \mathcal{B}_0 \to \mathcal{B}$ be given by $\Phi(B) = \lim_{n\to\mathcal{F}} B$. □

Proofs of Theorem 3.3 from Proposition 3.4. To prove Theorem 3.3 note that $(G, \mathcal{P}(G), d_{\mathcal{F}}, (L_g)_{g\in G})$, where $L_g : G \to G$ is given by $h \mapsto gh$, is a finitely additive measure preserving system. So we can apply Proposition 3.4to obtain a m.p.s. $(X, \mathcal{B}, \mu, T)$ and $\Phi : \mathcal{P}(G) \to \mathcal{B}$ such that, letting $B = \Phi(A)$, we have

$$\begin{aligned}\mu(T_{h_1}(B) \cap \cdots \cap T_{h_k}(B)) &= \mu(T_{h_1}(\Phi(A)) \cap \cdots \cap T_{h_k}(\Phi(A))) = \mu(\Phi(h_1 A) \cap \cdots \cap \Phi(h_k A)) \\ &= \mu(\Phi(h_1 A \cap \cdots \cap h_k A)) = d_{F,\mathcal{F}}(h_1 A \cap \cdots \cap h_k A).\end{aligned}$$ □

We note that more general versions of the FCP, like a version for actions of amenable groups given in [BM13, Theorem 2.8], also follow[5] from the same proof of Theorem 3.1, taking an ultrafilter density and then transforming it via Proposition 3.4 into a probability measure.

[3] This result seems to be well-known, but we did not find a similar statement in the literature.

[4] We stated the proposition for groups to simplify notation, but it also works for semigroups (e.g. noninvertible measure preserving actions), with the property $\Phi(S_g^{-1}(A)) = T_g^{-1}(\Phi(A))$ instead of Item 3.

[5] The m.p.s. we obtain need not be Borel in a compact metric space. That is not hard to fix, but we will not do it in this document.

4 Reverse Furstenberg correspondence

A natural question arising from Theorem 3.2 is whether we can find a Furstenberg correspondence principle in the opposite direction. That is, for any given measurable m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ can we find a set $A \subseteq G$ which satisfies Equation (8) ? The answer is positive:

Theorem 4.1 ([R2, Theorem 1.16])**.** *Let G be a countably infinite amenable group with a Følner sequence $(F_N)_N$. For every m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and every $B \in \mathcal{B}$ there exists a subset $A \subseteq G$ such that, for all $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in G$ we have*

$$d_F\left(h_1 A \cap \cdots \cap h_k A\right) = \mu\left(T_{h_1} B \cap \cdots \cap T_{h_k} B\right). \tag{10}$$

Reciprocally, for any $A \subseteq G$ there is a m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ satisfying Equation (10) for all $k, h_1, \ldots, h_k$ as above whenever $d_F\left(h_1 A \cap \cdots \cap h_k A\right)$ exists.

Our proof of Theorem 4.1 needs a result of Downarowicz, Huczek and Zhang about tilings of amenable groups, [DHZ, Theorem 5.2]; see a statement with a brief review of the definitions involved in [R2, 2.6-2.10]. Farhangi and Tucker-Drob give a different proof of Theorem 4.1 in their paper [FT] (uploaded to arXiv simultaneously to [R2]), but they still need [DHZ, Theorem 5.2].

Theorem 4.1 is a special case of a more general theorem relating Cesàro averages of sequences with integrals of functions:

Theorem 4.2. *Let G be a countably infinite amenable group with a Følner sequence $(F_N)_N$ and let $D \subseteq \mathbb{C}$ be compact. Then for any m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and any measurable function $f : X \to D$ there is a sequence $(z_g)_{g\in G}$ of complex numbers in D such that, for all $j \in \mathbb{N}$, $h_1, \ldots, h_j \in G$ and all continuous functions $p : D^j \to \mathbb{C}$,*

$$\lim_N \frac{1}{|F_N|} \sum_{g\in F_N} p(z_{h_1 g}, \ldots, z_{h_j g}) = \int_X p(f(T_{h_1} x), \ldots, f(T_{h_j} x)) d\mu. \tag{11}$$

Conversely, given a sequence $(z_g)_{g\in G}$ in D there is a m.p.s. $(X, \mathcal{B}, \mu, T)$ and a measurable function $f : X \to D$ such that, for all $j \in \mathbb{N}, h_1, \ldots, h_j \in G$ and $p : D^j \to \mathbb{C}$ continuous, Equation (11) holds if the limit in the left hand side exists.

In some cases one may adapt Theorem 4.2 to unbounded functions f, see [FT, Theorem 3.3].

4.1 Applications

We now state some non-trivial consequences of Theorem 4.2 obtained in [R2].

4.1.1 'White noise'

Proposition 4.3 (White noise in $\mathbb{S}^1$, see [R2, Proposition 3.5])**.** *Let G be a countably infinite amenable group with a Følner sequence $(F_N)_N$. Then there exists a sequence $(z_g)_{g\in G}$ in $\mathbb{S}^1$ such that, for all distinct $h_1, \ldots, h_n \in G$ and for all $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ not all 0, we have*

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{g\in F_N} z_{h_1 g}^{\alpha_1} \cdots z_{h_n g}^{\alpha_n} = 0. \tag{12}$$

Remark 4.4. There is one very important case where Proposition 4.3 can be proved more easily: if the Følner sequence $(F_N)_N$ does not grow very slowly (e.g. if $\sum_n |F_N|^\alpha < \infty$ for all $\alpha \in (0, 1)$)

In this case, suppose one chooses the elements of a sequence $(z_g)_{g\in G}$ randomly and independently, according to the uniform distribution in $\mathbb{S}^1$. One can check that Equation (12) will be satisfied with probability 1. This strategy does not work in the general case, but we still use random sequences, hence the name 'white noise' (in probability theory, white noise refers to a random process where each sample is uncorrelated with any other sample.)

4.1.2 Turning $[0,1]$-valued functions into sets

A (non-trivial) consequence of Theorem 4.1 is the following:

Proposition 4.5 ([R2, Corollary 3.7])**.** *Let G be a countably infinite amenable group. For every m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and every measurable $f : X \to [0,1]$ there exists a m.p.s. $(Y, \mathcal{C}, \nu, (S_g)_{g\in G})$ and $A \in \mathcal{C}$ such that for all $k \in \mathbb{N}$ and all distinct $h_1, \dots, h_k \in G$ we have*

$$\nu\left(S_{h_1}^{-1}A \cap \dots \cap S_{h_k}^{-1}A\right) = \int_X f(T_{h_1}(x)) \cdots f(T_{h_k}(x)) d\mu. \quad \square$$

The key fact used in Proposition 4.5 is that $[0,1]$ is the convex closure of $\{0,1\}$; see [R2, Proposition 3.5] for a more general statement.

Question 1. Is Proposition 4.5 true for countable, non-amenable groups?

4.1.3 Answers to some questions in the literature

Van der Corput sets. The original motivation for developing Theorem 4.2 was to answer questions from Bergelson and Lesigne about vdC sets, which we define below.

Definition 4.6. Let $F = (F_N)_{N\in\mathbb{N}}$ be a Følner sequence in a countable amenable group G. We say that a sequence $(x_g)_{g\in G}$ in $\mathbb{T}$ is *F-u.d. mod* 1 if for any continuous function $f : \mathbb{T} \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{g\in F_N} f(x_g) = \int_{\mathbb{T}} f dm.$$

Definition 4.7 (cf. [BL, Page 44])**.** Let $F = (F_N)_{N\in\mathbb{N}}$ be a Følner sequence in a countable amenable group G. We say a subset H of G is F-vdC if any sequence $(x_g)_{g\in G}$ in $\mathbb{T}$ such that $(x_{hg} - x_g)_{g\in G}$ is F-u.d. mod 1 for all $h \in H$, is itself F-u.d. mod 1.

In [BL, Page 44] it is asked whether, given $H \subseteq G$ and two Følner sequences F_1, F_2 in G, H is F_1-vdC iff it is F_2-vdC. The answer is positive:

Theorem 4.8 ([R2, Theorem 1.5])**.** *Let G be a countably infinite amenable group with a Følner sequence $F = (F_N)_N$. A set $H \subseteq G$ is F-vdC in G if and only if for any m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and for any function $f \in L^\infty(\mu)$,*

$$\int_X f(T_h(x)) \cdot \overline{f(x)} d\mu(x) = 0 \textit{ for all } h \in H \text{ implies } \int_X f d\mu = 0.$$

It is also proved in [R2], using Theorem 4.2, that all nice vdC sets in amenable groups are sets of nice recurrence (see [R2, Pages 4-5] for definitions and context). We also mention that the paper [FS], uploaded to arXiv around a month before [R2], had its questions 1 and 2 answered in the positive by Theorem 4.1.

As a last application of Theorem 4.1 we state Theorem 4.9 below, which answers a question asked in [BF] after Remark 3.6. The question is whether for all countable amenable groups G and all Følner sequences F in G there is a set $E \subseteq G$ such that $\overline{d}_F(E) > 0$ but for all finite $A \subseteq G$, $\overline{d}_F\left(\cup_{g\in A} g^{-1}E\right) < \frac{3}{4}$.

Theorem 4.9 ([R2, Theorem 1.17]). *Let G be a countably infinite amenable group with a Følner sequence $F = (F_N)_N$. Then there is $E \subseteq G$ such that, for all finite $\varnothing \neq A \subseteq G$,*

$$d_F(E) = d_F\left(\cup_{g\in A} gE\right) = \frac{1}{2}.$$

4.2 RFC and uniform density

One may try to find a version of Theorem 4.1 for uniform density: Is it true that for every m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and $B \in \mathcal{B}$ there is a subset A of G such that for all $k \in \mathbb{N}$ and $h_1, \dots, h_k \in G$ we have

$$d_u\left(h_1 A \cap \cdots \cap h_k A\right) = \mu\left(T_{h_1} B \cap \cdots \cap T_{h_k} B\right)? \tag{13}$$

If true, this would of course imply Theorem 4.1. But it is false: it is not hard to check that there is no set $E \subseteq \mathbb{Z}$ such that $d_u(E) = d_u(E \cap E + 1) = \frac{1}{2}$. But obviously, considering a m.p.s. $(X, \mathcal{B}, \mu, T = \mathrm{Id}_X)$ and some $B \in \mathcal{B}$ with $\mu(B) = 1/2$, we have $\mu(B) = \mu(B \cap TB) = \frac{1}{2}$. But what if T is ergodic?

Question 2. Let G be an amenable group, let $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ be an ergodic m.p.s. and let $B \in \mathcal{B}$. Is there a subset A of G such that for all $k \in \mathbb{N}$ and $h_1, \dots, h_k \in G$ we have

$$d_u\left(h_1 A \cap \cdots \cap h_k A\right) = \mu\left(T_{h_1} B \cap \cdots \cap T_{h_k} B\right)? \tag{14}$$

A result in the positive direction is the following corollary of [BM22, Theorem 1.3].

Definition 4.10. Let X be a compact metric space. A continuous map $T : X \to X$ is said to be uniquely ergodic if there is only one T-invariant Borel probability measure on X.

Proposition 4.11. *If X is compact, metric, $T : X \to X$ is uniquely ergodic (with T-invariant measure μ) and $B \subseteq X$ satisfies $\mu(\partial B) = 0$. Then $\forall x \in X$ the set $A := \{n \in \mathbb{N}; T^n x \in B\}$ satisfies*

$$d_u(A \cap (A - n)) = \mu(B \cap T^{-n} B) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{15}$$

5 Sets of differences and sets of returns

We recall some standard definitions. Let $A \subseteq \mathbb{Z}$, $(X, \mathcal{B}, \mu, T)$ an invertible m.p.s. and $B \in \mathcal{B}$.

- The *set of differences* of A is $A - A = \{a - b; a, b \in A\}$.
- The *set of density returns* of A is $\mathrm{R}_{\overline{d}}(A) := \{n \in \mathbb{Z}; \overline{d}(A \cap (A - n)) > 0\}$.
- The *set of returns* of B is $\mathrm{R}_T(B) := \{n \in \mathbb{Z}; \mu(T^{-n} B \cap B) > 0\}$.

These three types of sets are equivalent, in the sense described by the following propositions.

Remark 5.1. One can define $A - A$ and $\mathrm{R}_{\overline{d}}(A)$ for any subset A of an amenable group, and $\mathrm{R}_{(T_g)_{g\in G}}(B)$ for any measure preserving action of a group G. Similarly, other notions of density instead of $\overline{d}$ (e.g. $d_F, \overline{d_F}, d_{\mathcal{F}}$, etc) lead to different notions of sets of density returns. Note also that $\mathrm{R}_{\overline{d}}(A) \subseteq A - A$, because for all $n \in \mathbb{Z}$, $\overline{d}(A \cap A - n) > 0$ implies that $n \in A - A$.

Proposition 5.2. *For every set $A \subseteq \mathbb{Z}$ there is an invertible m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ with $\mu(B) \leq \overline{d}(A)$ and $\mathrm{R}_T(B) = \mathrm{R}_{\overline{d}}(A)$. Reciprocally, for every invertible m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ there is some subset A of $\mathbb{Z}$ with $d(A) = \mu(B)$ and $\mathrm{R}_T(B) = \mathrm{R}_{\overline{d}}(A)$.*

Proof. The second statement is immediately implied by Theorem 4.1. For the first part, for each $n \in R := \mathrm{R}_{\overline{d}}(A)$ consider an ultrafilter $\mathcal{F}_n$ such that $d_{\mathcal{F}_n}(A \cap A - n) > 0$, and let $(X_n, \mathcal{B}_n, \mu_n, T_n), B_n$ be obtained by applying Theorem 3.3 to $A, \mathcal{F}_n$ and $F_N = \{1, \ldots, N\}$. Now consider the measure preserving system $(X, \mathcal{B}, \mu, T)$, where $X = \sqcup_{n \in R} X_n$, $\mathcal{B} = \{Y \in \mathcal{P}(X); Y \cap X_n \in \mathcal{B}_n$ for all $n \in R\}$, $\mu(Y) = \sum_{n \in R} \lambda_n \mu_n(Y \cap X_n)$ (where λ_n are positive numbers such that $\sum_{n \in R} \lambda_n = 1$) and $T(x) = T_n(x)$ for all $x \in X_n \subseteq X$. Then the set $B = \cup_{n \in R} B_n$ satisfies what we want. □

Proposition 5.3. *For every set $A \subseteq \mathbb{Z}$ there is a m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ with $\mu(B) = \overline{d}(A)$ and $\mathrm{R}_T(B) \subseteq A - A$. Reciprocally, for every m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ there is some subset A of $\mathbb{Z}$ with $\overline{d}(A) = \mu(B)$ and $A - A \subseteq \mathrm{R}_T(B)$.*

Proof. The first part is a direct consequence of Theorem 3.1.

For the second part, note that by removing an adequate null set from X[6], we can assume that for all $n \in \mathbb{N}$ $B \cap T^{-n} B$ either has positive measure or is empty. Now consider the function $f : X \to [0, 1]$; $f(x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{T^{-n}B}(x)$. By Fatou's lemma, we have

$$\int_X f d\mu \geq \limsup_N \int_X \frac{1}{N} \sum_{n=1}^{N} \chi_{T^{-n}B} = \mu(B). \tag{16}$$

Thus, there is some $x_0 \in X$ such that $f(x_0) \geq \mu(B)$, so the set $A := \{n \in \mathbb{N}; x \in T^{-n} B\}$ has upper density $\geq \mu(B)$ and, satisfies $A \cap A - n = \varnothing$ for all $n \notin \mathrm{R}_T(B)$, due to the fact that $B \cap T^{-n} B = \varnothing$ for all $n \notin \mathrm{R}_T(B)$. □

Remark 5.4. Proposition 5.2 and Proposition 5.3 imply that the following three statements are equivalent for a set $E \subseteq \mathbb{Z}$ (sets E satisfying these properties are called *sets of recurrence*):

- E intersects $\mathrm{R}(B)$ for all m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ with $\mu(B) > 0$.
- E intersects $\mathrm{R}_{\overline{d}}(A)$ for all $A \subseteq \mathbb{Z}$ with $\overline{d}(A) > 0$.
- E intersects $A - A$ for all $A \subseteq \mathbb{Z}$ with $\overline{d}(A) > 0$.

6 Sumsets in sets of differences

6.1 $B + B \subseteq A - A$

Given a group G and $A \subseteq G$ (or $A \subseteq \mathbb{N}$[7]), we define $A + A = \{a + b; a, b \in A\}$. Our starting point is the following result of Bergelson, [Be84, Corollary 3.1.2]:

Theorem 6.1. *If $A \subseteq \mathbb{Z}$ has positive upper Banach density, then there exists a set of positive upper density $B \subseteq \mathbb{Z}$ such that $B + B \subseteq A - A$.*

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[6]Specifically, remove from X all null sets of form $T^{-n} A \cap T^{-m} A$ for $n, m \in \mathbb{N}$. One can check that the resulting set $X' \subseteq X$ satisfies $T(X') \subseteq X'$

[7]One can define these concepts for semigroups, but we will not need that.

The arguments from [Be84] allow one to obtain in Theorem 6.1 a set B such that $\overline{d}(B) \geq \frac{d^*(A)^4}{4}$. But a slightly different argument (found e.g. in the proof of [BR, Theorem 3.1]) gives us a higher value of $\overline{d}(B)$:

Theorem 6.2. *If $A \subseteq \mathbb{Z}$ has upper Banach density $a > 0$, then there exists a set $B \subseteq \mathbb{N}$ such that $\overline{d}(B) \geq a^2$ and $B + B \subseteq A - A$.*

Question 3. Let $a > 0$. What is the biggest possible value δ_a such that, if $A \subseteq \mathbb{Z}$ has upper Banach density a, then there exists $B \subseteq \mathbb{N}$ such that $\overline{d}(B) \geq \delta_a$ and $B + B \subseteq A - A$?

The key to proving Theorem 6.2 is the following classical result about ergodic averages.

Proposition 6.3. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., let $B \in \mathcal{B}$ satisfy $\mu(B) = a > 0$. Then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(B \cap T^{-n}B) \geq a^2.$$

Proof. Let $L_T \subseteq L^2(X, \mu)$ be the space of functions such that $f \circ T = f$, and let π_T, π_1 be orthogonal projections from $L^2(X, \mu)$ to L_T and $\mathbb{C}1$ respectively. By the mean ergodic theorem we have the L^2-limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{T^{-n}B} = \pi_T(\chi_B). \tag{17}$$

So taking scalar products with χ_B at both sides, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(B \cap T^{-n}B) = \langle \chi_B, \pi_T(\chi_B) \rangle = \|\pi_T(\chi_B)\|^2 \geq \|\pi_1(\chi_B)\|^2 = \|\mu(B)1\|^2 = \mu(B)^2, \tag{18}$$

where the inequality follows from the fact that $\mathbb{C}1 \subseteq L_T$. □

Let us now see how Proposition 6.3 implies Theorem 6.2. In the proof of the implication we will need a more general version of the intersectivity lemma (for the original see [Be84, Page 2]).

Proposition 6.4 (Intersectivity lemma)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space and let $(A_n)_n$ be a sequence of measurable subsets of X and let $\lambda := \limsup_N \frac{1}{N} \sum_{n=1}^{N} \mu(A_n)$. Then there is a set $E \subseteq \mathbb{N}$ with $\overline{d}(E) = \lambda$ such that, for any finite subset F of E, $\cap_{n\in F} A_n$ has positive measure.*

Proof. We can assume by removing from X an adequate null set that all finite intersections of the sets A_n are either empty or have positive measure (specifically, remove from X the empty intersections and their inverse images by powers of T). So it will be enough to find a point $x \in E$ such that the set $E_x := \{n \in \mathbb{N}; x \in A_n\}$ satisfies $\overline{d}(E) = \lambda$. Equivalently, we need to prove that the function $f : X \to [0, 1]$; $f(x) = \limsup_N \frac{1}{N} \sum_{n=1}^{N} \chi_{A_n}(x)$, satisfies $f(x) \geq \lambda$ for some x. But this is true, because by Fatou's lemma we have

$$\int_X f d\mu \geq \limsup_{N\to\infty} \int_X \frac{1}{N} \sum_{n=1}^{N} \chi_{A_n} d\mu = \limsup_N \frac{1}{N} \sum_{n=1}^{N} \mu(A_n) = \lambda.$$ □

The following proof is similar to the one of [Be84, Theorem 3.1].

Proof of Theorem 6.2. By the Furstenberg correspondence principle we can choose some invertible m.p.s. $(X,\mathcal{B},\mu,T)$ and $C\in\mathcal{B}$ with $\mu(C)=d^*(A)$ such that $\mu(C\cap T^{-n}C)\leq d^*(A\cap(A-n))$ for all $n\in\mathbb{N}$. Note that by Proposition 6.3 we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(T^{-n}C\cap T^nC)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(C\cap(T^2)^{-n}C)\geq\mu(C)^2=(d^*(A))^2.$$

So by Proposition 6.4 there is a set $B\subseteq\mathbb{N}$ with $\overline{d}(B)\geq(d^*(A))^2$ such that, for all $n,m\in B$ we have

$$(T^{-n}C\cap T^nC)\cap(T^{-m}C\cap T^mC)\neq\varnothing.$$

Thus $\mu(T^{-n}C\cap T^mC)>0$, so $d^*(A\cap A-(m+n))\geq\mu(T^{-n-m}C\cap C)>0$, so $m+n\in A-A$. □

Using other multiple recurrence theorems we can prove more versions of this result:

Proposition 6.5. *Let $A\subseteq\mathbb{Z}$ satisfy $d^*(A)>0$. Then there exists $B\subseteq\mathbb{Z}$ satisfying $\overline{d}(B)>0$ and $(B+B)\cup(B^2+B^7)\subseteq A-A$, where by B^2+B^7 we mean $\{a^2+b^7;a,b\in B\}$.*

Proof. The same proof as Theorem 6.2, but instead of $\lim_N\frac{1}{N}\sum_{n=1}^{N}\mu(T^nC\cap T^{-n}C)>0$ we use that, by [BL, Theorem A],

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(T^{-n^2}C\cap T^{-n}C\cap T^nC\cap T^{n^7}C)>0.$$ □

6.1.1 Abelian groups

For abelian groups, Theorem 6.2 generalizes without further complication:

Proposition 6.6. *Let $(G,\cdot)$ be a countable abelian group and let $A\subseteq G$ satisfy $d^*(A)=a>0$. Then for any Følner sequence $F=(F_N)_N$ there is some $B\subseteq G$ with $\overline{d}_F(B)\geq a^2$ and $BB\subseteq A^{-1}A$.*

Proof. The proof is the same as for $\mathbb{Z}$. Consider by Furstenberg correspondence a m.p.s. $(X,\mathcal{B},\mu,(T_g)_{g\in G})$ and some $C\in\mathcal{B}$ with $\mu(C\cap T_gC)\leq\overline{d}(A\cap(gA))$ for all $g\in G$. Remove an adequate null set from X so that intersections of sets of form T_gC are either empty or have positive measure.

Now for all $g\in G$ let $f_g=\chi_{T_gC\cap T_{g^{-1}}C}$, and let $f:X\to[0,1]$ be given by

$$f(x)=\limsup_{N\to\infty}\frac{1}{|F_N|}\sum_{g\in F_N}f_g(x). \tag{19}$$

Then we have, letting $S_g=T_{g^2}$ for all $g\in G$,

$$\int_X fd\mu\geq\limsup_N\frac{1}{|F_N|}\sum_{g\in F_N}\int_X f_g(x)dx \tag{20}$$

$$=\limsup_N\frac{1}{|F_N|}\sum_{g\in F_N}\mu(T_gC\cap T_{g^{-1}}C)dx \tag{21}$$

$$=\limsup_N\frac{1}{|F_N|}\sum_{g\in F_N}\mu(S_gC\cap C)dx\geq a^2. \tag{22}$$

so $f(x)\geq a^2$ for some $x\in X$, so the set $B:=\{g\in G;x\in T_{-g}C\cap T_gC\}$ has upper density $\geq a^2$. For all $b,b'\in B$ we have $x\in T_{-b}C\cap T_{b'}C$, so $\mu(C\cap T_{b+b'}C)>0$, so $b+b'\in A-A$, as we wanted. □

Question 4. Is Proposition 6.6 true for arbitrary amenable groups G and Følner sequences F?

By the same proof as in the abelian case, an affirmative answer to Question 4 would follow if for any m.p.s. $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ and any $B \in \mathcal{B}$ with $\mu(B) > 0$ we had

$$\lim_N \frac{1}{|F_N|} \sum_{g\in F_N} \mu(T_g^{-1}C \cap T_g C) = \lim_N \frac{1}{|F_N|} \sum_{g\in F_N} \mu(C \cap T_{g^2} C) > 0.$$

6.2 $f(B) + f(B) \subseteq A - A$ for general sequences f

In the last lines of [Be84] there is a plausible statement that we currently do not know how to prove:

Statement 6.7. *Given $A \subseteq \mathbb{N}$ with $d^*(A) = a > 0$, there is $B \subseteq \mathbb{N}$ with $\overline{d}(B) \geq \frac{a^4}{4}$ such that $b_1^2 + b_2^2 \in A - A$ for all $b_1, b_2 \in B$.*

Definition 6.8. Let $f : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers, and let $a > 0$. We denote by $\delta(f, a) > 0$ the biggest number such that for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $C \in \mathcal{B}$ with $\mu(C) \geq a$ we have

$$\overline{\lim}_N \frac{1}{N} \sum_{n=1}^{N} \mu(C \cap T^{-f(n)} C) \geq \delta. \tag{23}$$

The same proof of Theorem 6.2 gives the following more general statement:

Proposition 6.9. *Let $f : \mathbb{N} \to \mathbb{N}$ and let $A \subseteq \mathbb{N}$ with $d^*(A) = a > 0$. Then there is $B \subseteq \mathbb{N}$ with $\overline{d}(B) \geq \delta(f, a)$ such that $f(i) + f(j) \in A - A$ for all $i, j \in B$.* □

Remark 6.10. If $f(\mathbb{N})$ is not a set of recurrence, then we have $\delta(f, a) = 0$ for small enough $a > 0$. The reciprocal implication is not true: Forrest proved in [Fo] the existence of a set of recurrence $R \subseteq \mathbb{N}$ and a set A with $\mu(A) = 1/2$ but $\lim_{r\in R, r\to\infty} \mu(A \cap T^{-r} A) = 0$.

6.2.1 Ergodic sequences

We now recall a family of sequences f for which we can choose $\delta(f, a) = a^2$ for all a in Proposition 6.9.

Definition 6.11. A sequence $f : \mathbb{N} \to \mathbb{N}$ is *ergodic* if for all $z \in \mathbb{S}^1 \setminus \{1\}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z^{f(n)} = 0.$$

Let us see some examples of (non-)ergodic sequences.

1. The sequence $f(n) = n$ is ergodic.

2. The sequence $f(n) = n^2$ is not ergodic; it satisfies the definition of ergodicity for all $z \in \mathbb{S}^1$ that are not roots of unity, but it fails for some roots of unity. More generally, the sequence $f(n) = p(n)$ is not ergodic for any $p \in \mathbb{Z}[n]$ with degree ≥ 2. We study the case $f(n) = n^2$ in Section 6.2.2.

3. The sequence $f(n) = \lfloor n^c \rfloor$ is ergodic for all $c > 0, c \notin \mathbb{Z}$.

4. The sequence $f(n) =$ sum of digits of n in base 10 is ergodic.

The following is well-known, it is a corollary of [BE, Theorem 1]

Proposition 6.12. *If a sequence $f : \mathbb{N} \to \mathbb{N}$ is ergodic, then for all m.p.s.s $(X, \mathcal{B}, \mu, T)$ and for all $C \in \mathcal{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(C \cap T^{-f(n)}C) \geq \mu(C)^2$$

Thus for ergodic sequences f we have $\delta(f, a) = a^2$ for all $a \in [0, 1]$.

Proof. Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $C \in \mathcal{B}$ satisfy $\mu(C) = a$. We may assume that T is invertible. The sequence $(a_n)_{n\in\mathbb{Z}}$ given by $a_n = \mu(C \cap T^n C)$ is positive definite, so by Herglotz-Bochner's theorem there is a Borel probability measure μ in $\mathbb{S}^1$ such that for all $n \in \mathbb{Z}$,

$$\mu(C \cap T^n C) = \int_{\mathbb{S}^1} z^n d\mu.$$

Thus, by the dominated convergence theorem and as f is ergodic, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(C \cap T^{-f(n)}C) = \int_{\mathbb{S}^1} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z^{-f(n)} d\mu(z) = \mu(\{1\}).$$

Similarly we can deduce that $\lim_N \frac{1}{N} \sum_{n=1}^{N} \mu(C \cap T^{-n}C) = \mu(\{1\})$, so by Proposition 6.3 we obtain $\mu(\{1\}) \geq \mu(C)^2$. Thus, $\delta(f, a) \geq a^2$. The fact that $\delta(f, a) \leq a^2$ is because, given $a \in [0, 1]$, it is easy to construct a m.p.s. $(X, \mathcal{B}, \mu, T)$ and $C \in \mathcal{B}$ such that $\mu(C) = a$ and $\mu(C \cap T^{-n}C) = a^2$ for all $n \in \mathbb{N}$. ☐

Corollary 6.13. *Let $f : \mathbb{N} \to \mathbb{N}$ be an ergodic sequence of natural numbers. If $A \subseteq \mathbb{Z}$ has positive upper Banach density, then there exists a set $B \subseteq \mathbb{N}$ such that $\overline{d}(B) = d^*(A)^2$ and $f(b)+f(b') \in A-A$ for all $b, b' \in B$.*

6.2.2 $p(B) + p(B) \subseteq A - A$

We now address the question of whether, given $A \subseteq \mathbb{N}$ with $d^*(A) > 0$ and a polynomial $p \in \mathbb{Z}[n]$ of degree ≥ 2, we can find a set $B \subseteq \mathbb{N}$ with $\overline{d}(B) > 0$ such that $p(b) + p(b') \in A - A$ for all $b, b' \in B$. That can be reduced via Proposition 6.9 to the question of, given a polynomial $p : \mathbb{N} \to \mathbb{N}$ and $a > 0$, is there a positive constant $\delta(p, a)$ (as in Definition 6.8) such that, for all invertible m.p.s.s $(X, \mathcal{B}, \mu, T)$ and for all $C \in \mathcal{B}$ with $\mu(C) \geq a$, we have

$$\limsup_{N} \frac{1}{N} \sum_{n=1}^{N} \mu\left(C \cap T^{p(n)}C\right) \geq \delta(p, a).$$

This question is only interesting for *intersective polynomials*, that is, polynomials $p \in \mathbb{Z}[n]$[8] such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $n|p(m)$. It is easy to see that if a polynomial p is not intersective, then $p(\mathbb{Z})$ is not a set of recurrence, so by Remark 6.10 we have $\delta(p, a) = 0$ for small enough a. For intersective polynomials p we have $\delta(p, a) > 0$, as shown in [BR].

Question 5. Given a fixed intersective polynomial $p \in \mathbb{Z}[n]$, what is the growth of $\delta(p, a)$ when $a \to 0$? Does it decrease polynomially with a?

[8] We could also consider polynomials in $\mathbb{Q}[n]$ that map $\mathbb{Z}$ to $\mathbb{Z}$, for example $\frac{n(n+1)}{2}$.

In the rest of the section we explain the upper and lower bounds we have obtained for Question 5. The case of polynomials is not as nice as the case of ergodic sequences: let $A = \{0, 2\} \subseteq \mathbb{Z}_5 := \mathbb{Z}/5\mathbb{Z}$. Then, letting μ be the uniform measure in $\mathbb{Z}_5$ and letting $T : \mathbb{Z}_5 \to \mathbb{Z}_5; n \mapsto n+1$, we have

$$\frac{2}{25} = \lim_N \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) < \mu(A)^2 = \frac{4}{25}.$$

So for $a = 0.4$ we already have[9] $\delta(n^2, a) < a^2$. More generally, we use a construction of Ruzsa [Ru] involving finite rotations to prove the following (see Appendix B for the proof)

Proposition 6.14. *There are arbitrarily small values of $a > 0$ such that $\delta(n^2, a) < a^{2.87555}$.*

The best lower bound we found for $\delta(n^2, a)$ is very far from the upper bounds; in fact, it is not a priori obvious that $\delta(n^2, a) > 0$ for all a, that is, that for all $a > 0$ there is a constant $\delta > 0$ such that, for all m.p.s.s $(X, \mathcal{B}, \mu, T)$ and for all $A \in \mathcal{B}$ with $\mu(A) > 0$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) > \delta.$$

Proposition 6.15. *For small enough $a > 0$, we have $\delta(n^2, a) \geq e^{-1/\sqrt{a}}$.*

6.3 $B \times B \subseteq A - A$

Most results in previous sections can be given for cartesian products $B \times B$ instead of sums $B+B$, if A is a positive density subset of $\mathbb{N}\times\mathbb{N}$. More concretely, we have the following analog to Proposition 6.9:

Proposition 6.16. *Let $f : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers, $A \subseteq \mathbb{Z}^2$ with $d^*(A) = a > 0$ and $\delta = \delta(f, a)$. Then there exists $B \subseteq \mathbb{N}$ such that $\overline{d}(B) \geq \delta$ and $(n, m) \in A - A$ for all $n, m \in B$.* □

Proof. By Theorem 3.1 there is a m.p.s. $(X, \mathcal{B}, \mu, (U_g)_{g\in\mathbb{Z}^2})$ and $C \in \mathcal{B}$ such that $\mu(C) = d^*(A) \geq a$ and $d^*(A \cap (A + (n, m))) \leq \mu(C \cap U_{(n,m)} C)$. Let $T = U_{(1,0)}$ and $S = U_{(0,1)}$. Now we use that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{f(n)} C \cap S^{-f(n)} C) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(C \cap (TS)^{f(n)} C) \geq \delta,$$

so by Proposition 6.4 there must be a set $B \subseteq \mathbb{N}$ with $\overline{d}(B) > \lambda$ such that, for all $n, m \in B$, $\mu\left(T^{f(n)} C \cap S^{-f(n)} C \cap T^{f(m)} C \cap S^{-f(m)} C\right) > 0$. Thus, in particular, $\mu\left(T^{f(n)} C \cap S^{-f(m)} C\right) > 0$, so $\mu(C \cap T^{f(n)} S^{f(m)} C) > 0$, that is, $\mu(C \cap U_{(n,m)} C) > 0$. This implies by definition of C that $d^*(A \cap (A + (n, m))) > 0$, so $(n, m) \in A - A$, as we wanted. □

In [BR] an example is given of a set $A \subseteq \mathbb{Z}^3$ of positive density such that we do not have $B \times B \times B \subseteq A - A$ for any $B \subseteq \mathbb{Z}$ with positive upper density. However, the question for sums is still open:

Question 6. Given a set $A \subseteq \mathbb{Z}$ with $\overline{d}(A) > 0$, is there a set $B \subseteq \mathbb{Z}$ with $\overline{d}(B) > 0$ such that $B + B + B \subseteq A - A$?

In [BR] Bergelson and Ruzsa mention that they expect Question 6 to have a negative answer.

[9] We proceed to abuse notation by denoting by 'n^2' the polynomial $p : \mathbb{N} \to \mathbb{N}; p(n) = n^2$

6.4 $B \cdot B \subseteq A - A$

A question similar to the ones in previous sections is whether, if $A \subseteq \mathbb{N}$ satisfies $d^*(A) > 0$, we can achieve $BB \subseteq A - A$ (where $BB = \{b_1 b_2; b_1, b_2 \in B\}$) for some 'big' set B. The answer is positive if we only ask B to be infinite. In fact, many stronger results can be proven: for example, the following is a corollary of [BEHL, Theorem 2.3]:

Proposition 6.17. *If $A \subseteq \mathbb{N}$ satisfies $d^*(A) > 0$, there is some infinite set $B \subseteq \mathbb{N}$ such that*

$$(B + B) \cup BB \subseteq A - A.$$

However, sometimes we cannot obtain a set B such that $d^*(B) > 0$ but $BB \subseteq A - A$:

Proposition 6.18. *Let $A = \{n \in \mathbb{Z}; |\pi n|_{\mathbb{T}} < 0.1\}$ (where $\|x\|_{\mathbb{T}} = \min_{k \in \mathbb{Z}} |k - x|$). Then A is a set of returns, but there is no set $B \subseteq \mathbb{N}$ such that $d^*(B) > 0$ and $BB \subseteq A - A$.*

Proof. See Appendix B. □

6.5 $B + B \subseteq A - A$ and $BB \subseteq A^{-1}A$ in countable fields

Can we obtain a subset B of a countable field such that $B + B \subseteq A - A$ and $BB \subseteq A^{-1}A$ at the same time?

Definition 6.19. A *double Følner sequence* in a countable field $(Q, +, \cdot)$ is a $(Q, +)$-Følner sequence $(F_N)_N$ such that $(F_N \setminus \{0\})_N$ is a $(Q^*, \cdot)$-Følner sequence.

Example 6.20. An explicit example of a double Følner sequence $(F_N)_N$ in $\mathbb{Q}$ is given by $F_N = A_N \cdot B_N = \{ab; a \in A_n, b \in B_N\}$, where, if $(p_n)_n$ are the prime numbers,

$$A_N = \left\{-2^{2^N}, -2^{2^N} + 1, \dots, 2^{2^N}\right\},$$
$$B_N = \{2^{-i_1} 3^{-i_2} \cdots p_N^{-i_N}; i_1, \dots, i_N \in \{N, N+1, \dots, 2N\}\}.$$

Fact 6.21. All countable fields have double Følner sequences (see [BM13, Proposition 2.4]).

Question 7. Let $(F_N)_N$ be a double Følner sequence in an infinite countable field $(Q, +, \cdot)$, and let $A \subseteq Q$ satisfy $\overline{d}_F(A) > 0$. Is there any set $B \subseteq Q$ with $\overline{d}_F(B) > 0$ and such that $B + B \subseteq A - A$ and $BB \subseteq A^{-1}A$? Future Saúl: Yes! This is true.

Remark 6.22. In $\mathbb{N}$ there are no 'double Følner' sequences $F = (F_N)_N$ of finite subsets of $\mathbb{N}$ such that $\lim_N \frac{|F_N \cap (F_N - n)|}{|F_N|} = \lim_N \frac{|F_N \cap (F_N / n)|}{|F_N|} = 0$ for all $n \in \mathbb{N}$. If so, the set of odd numbers $O \subseteq \mathbb{N}$ would satisfy $d_F(O) = 1/2$ (due to additive invariance) but $d_F(O) = 0$ (due to multiplicative invariance). For additive Følner sequences, even for the classical $F_N = \{1, \dots, N\}$,Question 7 has a negative answer in $\mathbb{N}$, e.g. [Be35] Besicovitch proved that there are sets $A \subseteq \mathbb{N}$ such that $\overline{d}(A)$ is as close as we want to $\frac{1}{2}$ but $A \cap nA = \varnothing$ for all $n \in \mathbb{N}, n \geq 2$.

An condition sufficient for a positive answer to Question 7 is the following ergodic theorem involving the additive and multiplicative actions of Q:

Statement 6.23. *Let $(F_N)_N$ be a double Følner sequence in a countable field $(Q, +, \cdot)$. Let $(X, \mathcal{B}, \mu, (T^q)_{q \in Q})$ and $(Y, \mathcal{C}, \nu, (S^q)_{q \in Q^\times})$ be measure preserving systems, and let $B \in \mathcal{B}, C \in \mathcal{C}$ satisfy $\mu(B), \nu(C) > 0$. Then we have*

$$\limsup_N \frac{1}{|F_N|} \sum_{q \in F_N \setminus \{0\}} \mu(B \cap T^q B) \nu(C \cap S^q C) > 0. \tag{24}$$

A Loeb measures from scratch

Loeb measures allow us to construct a 'limit' of a sequence of measure-preserving systems. Gelfand representation can also be used to do the same, so we are just providing an alternative method. The existence of the constructions below is stated e.g. in [AB, Page 25].

This section aims to introduce limits of m.p.s.s using Loeb measures,

- Briefly and with proofs.
- Requiring no non-standard analysis background, just the basic properties of ultrafilters introduced in Section 1.

Fix a non-principal ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ in $\mathbb{N}$, and for each $n \in \mathbb{N}$ let $(X_n, \mathcal{B}_n, \mu_n)$ be a finitely additive probability measure preserving system.

Definition A.1 (Ultraproduct of sets)**.** The *ultraproduct* $X_\infty := \lim_{n\to\mathcal{F}} X_n$ is the quotient set $\frac{\prod_{n\in\mathbb{N}} X_n}{\sim}$, where $(x_n)_n \sim (y_n)_n$ iff $x_n = y_n$ for $\mathcal{F}$-almost all n. We denote by $[y_n]_n \in \lim_{n\to\mathcal{F}} X_n$ the class of an element $(y_n)_n$.

It follows from Definition 1.1 that $\sim$ is an equivalence relation.

Definition A.2 (Internal subsets)**.** Given a sequence of sets $A_n \subseteq X_n$, we denote $\lim_{n\to\mathcal{F}} A_n = \{[x_n]_n \in X_\infty; x_n \in A_n \text{ for } \mathcal{F}\text{-almost all } n\}$. Subsets of X_∞ which can be obtained this way are called *internal*.

Note that $\lim_{n\to\mathcal{F}} A_n = \lim_{n\to\mathcal{F}} B_n$ iff $A_n = B_n$ for $\mathcal{F}$-almost all n. Similarly, $\lim_{n\to\mathcal{F}} A_n = \varnothing$ iff $A_n = \varnothing$ for $\mathcal{F}$-almost all n. One can check the following.

Proposition A.3. *For all probability measure spaces $(X_n, \mathcal{B}_n, \mu_n)$ and $A_n, B_n \in \mathcal{B}_n$,*

1. $\lim_{n\to\mathcal{F}} A_n \cup B_n = (\lim_{n\to\mathcal{F}} A_n) \cup (\lim_{n\to\mathcal{F}} B_n)$.
2. $\lim_{n\to\mathcal{F}} A_n \cap B_n = (\lim_{n\to\mathcal{F}} A_n) \cap (\lim_{n\to\mathcal{F}} B_n)$.
3. $\lim_{n\to\mathcal{F}} A_n \setminus B_n = (\lim_{n\to\mathcal{F}} A_n) \setminus (\lim_{n\to\mathcal{F}} B_n)$.

Proof. We prove $\subseteq$ in 1. Let $[x_n]_n \in \lim_{n\to\mathcal{F}} A_n \cup B_n$, so that $x_n \in A_n \cup B_n$ for $\mathcal{F}$-almost all $n \in \mathbb{N}$. Thus, by Remark 1.2, either $x_n \in A_n$ for $\mathcal{F}$-almost all n or $x_n \in B_n$ for $\mathcal{F}$-almost all n. So $[x_n]_n \in \lim_{n\to\mathcal{F}} A_n \cup \lim_{n\to\mathcal{F}} B_n$. □

Thus, the family of internal subsets is an algebra of subsets of X_∞.

Proposition A.4. *For each $k \in \mathbb{N}$ let $A^k = \lim_{n\to\mathcal{F}} A_n^k$ be an internal subset of X_∞. If the sets A^k are nonempty and pairwise disjoint, then $A := \cup_{k\in\mathbb{N}} A_k$ is not internal.*

Proof. Suppose that $A = \lim_n A_n$ for some sets $A_n \subseteq X_n$. For each $k, n \in \mathbb{N}$ let

$$B_n^k := A_n \cap \left(A_n^k \setminus (A_n^1 \cup \cdots \cup A_n^{k-1})\right).$$

Then for each fixed k, the fact that $\varnothing \neq A \cap \left(A^k \setminus (A^1 \cup \cdots \cup A^{k-1})\right) = \lim_{n\to\mathcal{F}} B_n^k$ implies that $B_n^k \neq \varnothing$ for almost all n. We define a point $[x_n]_n \in A$ by letting $x_n \in X_n$ be some point of $B_n^{k_n}$, where k_n is given by:

1. If $B_n^j \neq \varnothing$ for finitely many values of j, let k_n be the maximum such j.

2. If $B_n^j \neq \varnothing$ for infinitely many j, let k_n satisfy $k_n > n$ and $B_n^{k_n} \neq \varnothing$.

There may be some values of n such that $B_n^k = \varnothing$ for all $k > n$. But that happens for $\mathcal{F}$-almost no n (as for all k we have that $B_n^k \neq \varnothing$ for $\mathcal{F}$-almost all n); we choose x_n however we want in that case.

We then have $[x_n]_n \in A$, because $x_n \in B_n^{k_n} \subseteq A_n$ for almost all n. But for each fixed value of k, $[x_n]_n \notin A_k$. Indeed, for $\mathcal{F}$-almost all n we have $k_n > k$ (this is obvious in Item 2, and in Item 1 it follows from the fact that $B_n^{k+1} \neq \varnothing$ for $\mathcal{F}$-almost all n), so $x_n \in B_n^{k_n} \subseteq A_n \setminus A_n^k$, so $x_n \notin A_n^k$. □

Now, let $\mathcal{A}_\infty$ be the algebra of all internal sets of the form $\lim_{n\to\mathcal{F}} A_n$, where $A_n \in \mathcal{B}_n$ for all n. And let $\mathcal{B}_\infty \subseteq \mathcal{P}(X_\infty)$ be the σ-algebra generated by $\mathcal{A}_\infty$.

Proposition A.5. *The map $\mu : \mathcal{A}_\infty \to [0,1]$; $\mu(\lim_{n\to\mathcal{F}} A_n) = \lim_{n\to\mathcal{F}} \mu_n(A_n)$ (see Proposition 1.3) is a pre-measure*[10]*, thus it extends to a probability measure $\mu : \mathcal{B}_\infty \to [0,1]$.*

Proof. Suppose we have disjoint sets $A^1, A^2, \dots$ in $\mathcal{A}_\infty$, with $A_i = \lim_{n\to\mathcal{F}} A_n^i$. Note that for $i \neq j$ we have $A_n^i \cap A_n^j = \varnothing$ for $\mathcal{F}$-almost all n. And suppose that $A := \cup_{k\in\mathbb{N}} A^k \in \mathcal{A}_\infty$, then by Proposition A.4 we have $A = \cup_{k=1}^K A^k$. So

$$\mu(A) = \mu\left(\cup_{k=1}^K A^k\right) = \mu\left(\lim_{n\to\mathcal{F}}(A_n^1 \cup \dots \cup A_n^K)\right) = \lim_{n\to\mathcal{F}} \mu_n(A_n^1 \cup \dots \cup A_n^K)$$
$$= \sum_{k=1}^K \lim_{n\to\mathcal{F}} \mu_n(A_n^k) = \sum_{k=1}^K \mu(A^k) = \sum_{k=1}^\infty \mu(A^k).$$

So as we wanted, μ is a premeasure. So by the Caratheodory extension theorem, it extends to a probability measure μ_∞ in $(X_\infty, \mathcal{B}_\infty)$. □

Remark A.6 (Limits of measure preserving actions)**.** If $T_n : X_n \to X_n$ is a measure preserving map for all $n \in \mathbb{N}$, then the map $T_\infty : X_\infty \to X_\infty; [x_n]_n \mapsto [Tx_n]_n$ satisfies $T^{-1} \lim_{n\to\mathcal{F}} A_n = \lim_{n\to\mathcal{F}} T_n^{-1} A_n$, so it is measurable and measure preserving. If $S_n : X_n \to X_n$ is also measure preserving, then we have $\lim_{n\to\mathcal{F}} T_n \circ S_n = (\lim_{n\to\mathcal{F}} T_n) \circ (\lim_{n\to\mathcal{F}} S_n)$, which allows us to define the limit of any sequence of measure preserving actions.

Definition A.7. Given a sequence $(X_n, \mathcal{B}_n, \mu_n, T_n)$ of finitely additive measure preserving systems, we denote by $\lim_{n\to\mathcal{F}}(X_n, \mathcal{B}_n, \mu_n, T_n)$ the measure preserving system $(X_\infty, \mathcal{B}_\infty, \mu_\infty, T_\infty)$, where $X_\infty, \mathcal{B}_\infty, \mu_\infty, T_\infty$ are constructed as in Proposition A.4 and Proposition A.5.

B Proofs of some statements from the main body

Proof of Proposition 6.14

We restate Proposition 6.14 for convenience:

Proposition B.1. *For arbitrarily small values of $a > 0$ there is a measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) = a$ and such that*

$$\lim_N \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n^2} A) = \mu(A)^{\frac{\ln(12/205^2)}{\ln(12/205)}} < \mu(A)^{2.87555}. \tag{25}$$

[10] That is, it satisfies that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets of $\mathcal{A}_\infty$ and $\cup_n A_n \in \mathcal{A}_\infty$, then $\mu(\cup_n A_n) = \sum_n \mu(A_n)$

We will need the following set $A \subseteq \mathbb{Z}_{205} := \frac{\mathbb{Z}}{205\mathbb{Z}}$ found by Beigel and Gasarch (see [BG]), which very slightly improves the original construction by Ruzsa:

Fact B.2 ([BG], Theorem 3.7)**.** 205 is squarefree, and the set

$$A := \{0, 2, 8, 14, 77, 79, 85, 96, 103, 109, 111, 181\} \subseteq \mathbb{Z}_{205} \tag{26}$$

has no square differences mod 205.

For each $k \in \mathbb{N}$ consider the m.p.s. $(\mathbb{Z}_{205^{2k}}, \mathcal{P}(\mathbb{Z}_{205^{2k}}), \mu, x \mapsto x+1)$, where μ is the uniform probability measure.

Now let A_k be the set of numbers $n \in \mathbb{Z}_{205^{2k}}$ of form $n = \sum_{j=0}^{2k-1} n_j 205^j$, where $n_j \in A$ if j is even and $n_j \in \{0, 1, \dots, 204\}$ if j is odd.

Then we have $\mu(A_k \cap (A_k - n^2)) = \varnothing$ whenever n is not a multiple of 205^k. Indeed, suppose we had some number $n = 205^l \cdot u$, where $l < k$ and $205 \not| u$, such that n^2 is of the form $a - b$, with $a, b \in A$. That cannot be true because in the expression $n^2 = \sum_{j=0}^{2k-1} n_j 205^j$, the coefficient n_{2l} is mod N equal to u^2, which is not a difference of two elements of A (here we used that 205 is squarefree, so $u^2 \neq 0$ mod 205).

So firstly, we have $\mu(A) = \frac{205^k \cdot 12^k}{205^{2k}} = \left(\frac{12}{205}\right)^k$. Secondly, $\mu(A \cap T^{-n^2} A)$ is $\mu(A)$ if n is a multiple of 205^k and 0 if not, so

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) = \frac{1}{205^k} \mu(A) = \left(\frac{12}{205^2}\right)^k = \mu(A)^{\frac{\ln(12/205^2)}{\ln(12/205)}} = \mu(A)^{2.87555\dots}.$$

Proof of Proposition 6.15

We first restate Proposition 6.15.

Proposition B.3. *There is some $\varepsilon > 0$ such that for all $a \in (0, \varepsilon)$ and for any m.p.s. $(X, \mathcal{B}, \mu, T)$ and $B \in \mathcal{B}$ with $\mu(B) \geq a$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(B \cap T^{-n^2} B) > e^{-1/\sqrt{a}}.$$

The following proof contains ideas from [TZ, Appendix B]. We will need the following:

Black Box B.4 (See [PSS])**.** *There is some $C > 0$ such that, for big enough N, any subset $A \subseteq \{1, \dots, N\}$ with $|A| \geq \frac{CN}{\ln(N)^{\frac{1}{4} \ln(\ln(\ln(\ln(N))))}}$ contains elements $a < b$ such that $b - a$ is a perfect square.*

In particular, if we fix some small enough $a > 0$ and we let $M_0 = \lfloor e^{\frac{1}{3\sqrt{a}}} \rfloor$, then for any $A \subseteq \{1, \dots, M_0\}\}$ with $\frac{|A|}{M_0} \geq \frac{a}{2}$ we have $A \cap (A - n^2) \neq \varnothing$ for some $n \in \mathbb{N}$. That is, for all $A \subseteq \{1, \dots, M_0\}$ with $\frac{|A|}{M_0} \geq \frac{a}{2}$ we have

$$\sum_{n=1}^{\lfloor \sqrt{M_0} \rfloor} |A \cap (A - n^2)| \geq 1.$$

This implies that, for any $M \geq M_0$ and $A \subseteq \{1, \dots, M\}$ such that $\frac{|A|}{M} \geq a$, we have

$$\sum_{n=1}^{\lfloor \sqrt{M_0} \rfloor} |A \cap (A - n^2)| \geq \frac{aM}{10M_0}.$$

(To prove this from the previous fact, divide the set $\{1,\ldots,M\}$ into $\{1,\ldots,M_0\},\{M_0+1,\ldots,2M_0\}$, $\ldots,\{(\lfloor M/M_0\rfloor+1)\,M_0,\ldots,\lfloor M/M_0\rfloor M_0\}$ and note that at least $\frac{a}{2}\lfloor M/M_0\rfloor$ of these sets have square differences). This in turn implies that any subset $E\subseteq\mathbb{Z}$ satisfying $\overline{d}(E)\geq a$ will satisfy $\sum_{n=1}^{\sqrt{M}}\overline{d}(E\cap(E-n^2))\geq\frac{a}{10M_0\lfloor\sqrt{M_0}\rfloor}$. So by Theorem 4.1 we have the following result.

Lemma B.5. *For any m.p.s.* $(X,\mathcal{B},\mu,S)$ *and* $B\subseteq\mathcal{B}$ *such that* $\mu(B)\geq a$*, we have*

$$\sum_{n=1}^{\lfloor\sqrt{M_0}\rfloor}\mu(A\cap S^{-n^2}A)\geq\frac{a}{10M_0\left\lfloor\sqrt{M_0}\right\rfloor}.$$

Now consider the sequence of real numbers $(a_n)_n$, with $a_n=\mu(B\cap T^{-n^2}B)$. We want to prove that $\lim_N\frac{1}{N}\sum_{n=1}^{N}a_n>e^{-1/\sqrt{a}}$. But for all $k\in\mathbb{N}$ we have, by Lemma B.5 applied to $S=T^k$, that

$$\frac{1}{\left\lfloor\sqrt{M_0}\right\rfloor}\sum_{n=1}^{\lfloor\sqrt{M_0}\rfloor}a_{kn}>\frac{a}{10M_0\left\lfloor\sqrt{M_0}\right\rfloor}.$$

So we are done by the following lemma relating Cesaro averages with finite averages, and using that $\frac{a}{10M_0\lfloor\sqrt{M_0}\rfloor^2}\geq e^{-1/\sqrt{a}}$ for small enough a.

Lemma B.6. *Let* $(a_n)_n$ *be a sequence in* $[0,1]$*. Suppose that there are* $K\in\mathbb{N}$ *and* $\delta>0$ *such that, for all* $k\in\mathbb{N}$*,*

$$\frac{1}{K}\sum_{n=1}^{K}a_{kn}\geq\delta.$$

Then we have

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n\geq\frac{\delta}{K}.^{11}$$

Proof. For all $N\in\mathbb{N}$ the sums $\sum_{n=1}^{K}a_{kn}$ are $\geq K\delta$ for all $k=1,\ldots,\lfloor N/K\rfloor$, and each summand in the expression $\frac{1}{N}\sum_{n=1}^{N}a_n$ is counted at most K times in the aforementioned sums. □

Proof of Proposition 6.18

We prove a stronger result:

Proposition B.7. *Let* $A=\{n\in\mathbb{Z};|\pi n|_{\mathbb{T}}<0.1\}$ *(where* $\|x\|_{\mathbb{T}}=\min_{k\in\mathbb{Z}}|k-x|$*). Then* A *is a set of returns, but there are no sets* $B,C\subseteq\mathbb{N}$ *such that* $d^*(B)>0,C$ *is infinite and* $BC\subseteq A$*.*

Note first that A is the set of returns of the interval $I=(0,0.1)\subseteq\mathbb{T}=\frac{\mathbb{R}}{\mathbb{Z}}\cong[0,1)$ by the map $T:\mathbb{T}\to\mathbb{T};Tx=x+\pi$.

Now, for each $X\subseteq\mathbb{T}$ and $n\in\mathbb{N}$, we denote $n^{-1}X=\{x\in\mathbb{T};cx\in\mathbb{T}\}$. Let μ be Lebesgue measure in $\mathbb{T}$. We now prove that (*) for any two measurable sets $B,C\subseteq\mathbb{T}$, we have $\lim_{n\to\infty}\mu(B\cap n^{-1}C)=\mu(B)\mu(C)$; this is obvious if $\mu(C)=0$ and if not, let $(a_n)_{n\in\mathbb{Z}},(b_n)_{n\in\mathbb{Z}}$ be the Fourier coefficients of

[11] One can obtain $\frac{\delta}{2\ln(K)}$ instead of $\frac{\delta}{K}$ in this equation by using a better argument, but we will not do so as the arguments in this proof are very suboptimal anyways.

χ_C, χ_B respectively, so that $\chi_C = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ in $L^2(\mathbb{T})$. In particular, $a_0 = \mu(C)$. Then the characteristic function $\chi_{c^{-1}C}$ has Fourier coefficients $(a_{nc})_{n\in\mathbb{Z}}$[12], so we have

$$\lim_{c\to\infty} \mu(B\cap c^{-1}C) = \lim_{c\to\infty} \sum_{n\in\mathbb{Z}} a_{cn} b_n = a_0 b_0 = \mu(C)\mu(B),$$

where in the middle equality we used that for all $i \in \mathbb{Z}\setminus\{0\}$ except 0, the i^{th} Fourier coefficient of χ_C goes to 0.

Also note that, for any subset X of $\mathbb{T}$ such that $\mu(\partial X) = 0$, the set $R_X := \{n \in \mathbb{N}; \pi n \in X\}$ satisfies $d_u(R_X) = \mu(X)$ (this follows e.g. from Proposition 4.11, as irrational rotations in $\mathbb{T}$ are classic examples of uniquely ergodic systems).

Now, the fact that $BC \subseteq A - A$ means that $\|bc\pi\|_{\mathbb{T}} < 0.1$ for all $b \in B, c \in C$. Equivalently, letting $J = \{x \in \mathbb{T}; \|x\|_{\mathbb{T}} < 0.1\}$, we have $\alpha B \subseteq J_\infty := \cap_{c\in C} c^{-1}J$.

But fact (*) above implies that, if $J_n = \cap_{c\in C; c\leq n} c^{-1}J$, then $\lim_{n\to\infty}\mu(J_n) = 0$. And $\alpha B \subseteq J_n$ for all n, so B is contained in the set $\{m \in \mathbb{N}; \pi m \in J_n\}$, which has uniform density $\to 0$ when $n \to \infty$. Thus, B cannot have positive upper density.

[12] For any function $f \in L^2(\mathbb{T})$ with Fourier coefficients $(a_n)_n$, the coefficients of $x \mapsto f(cx)$ are $(a_{cn})_n$; this is immediate for the character functions $f_k(x) = e^{2\pi i k x}$, so it is true for all functions.

M Metric geometry research

This section summarizes the findings in our research project under the guidance of Prof. Facundo Mémoli. More detailed explanations can be found in the introductions of [R1] and [R3]. This work falls within the area of metric geometry, focusing on Gromov-Hausdorff distances.

Let (X, d_X) be a metric space and let $A, B \subseteq X$ be nonempty. The *Hausdorff distance* between A and B is given by

$$d_{\mathrm{H}}^X(A,B) = \max\left(\sup_{a\in A} d_X(a,B), \sup_{b\in B} d_X(b,A)\right) \in [0,\infty].$$

Given two nonempty metric spaces (X, d_X) and (Y, d_Y), we write $(X, d_X) \cong (Y, d_Y)$, or just $X \cong Y$, whenever (X, d_X) and (Y, d_Y) are isometric. The *Gromov-Hausdorff (GH) distance* between X and Y is given by

$$d_{\mathrm{GH}}(X,Y) = \inf\{d_{\mathrm{H}}^Z(X',Y'); (Z,d_Z) \text{ metric space; } X',Y' \subseteq Z; X' \cong X; Y' \cong Y\}.$$

In particular, $d_{\mathrm{GH}}(X,Y) = 0$ if $X \cong Y$. Intuitively, GH distance measures 'how far two metric spaces are from being isometric'. Computing the exact value of the GH distance between metric spaces is quite challenging; there are only a few examples of pairs of spaces $(X, d_X), (Y, d_Y)$ for which we know $d_{\mathrm{GH}}(X,Y)$ (see the introduction of [R1] for a more detailed discussion and references).

[R1]: GH distance between spheres Of especial importance to our project was the paper [LMS], in which Lim, Mémoli and Smith tackle the problem of finding the GH distance between the spheres $\mathbb{S}^n$, $n = 1, 2, \ldots$ [13], with their geodesic metrics. They provided some upper and lower bounds for $d_{\mathrm{GH}}(\mathbb{S}^n, \mathbb{S}^m)$ for all $n, m \in \mathbb{N}$ and they gave exact values for the pairwise distances between $\mathbb{S}^1, \mathbb{S}^2$ and $\mathbb{S}^3$. A concrete case of [LMS, Theorem B] implies that

$$2d_{\mathrm{GH}}(\mathbb{S}^n, \mathbb{S}^{n+1}) \geq \zeta_n := \arccos\left(\frac{-1}{n+1}\right), \text{for all } n \in \mathbb{N}. \tag{27}$$

It was later proved in [ABC] that

$$d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^{2n}) \geq \frac{\pi n}{2n+1} \text{ and } d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^{2n+1}) \geq \frac{\pi n}{2n+1} \text{ for all } n \in \mathbb{N}. \tag{28}$$

Harrison and Jeffs proved in [HJ] (the version including Theorem M.2 below was uploaded to arXiv in July 2024) that Equation (28) is actually an equality:

Theorem M.1 ([HJ, Theorem 1.1])**.** *For all $n \geq 1$, $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^{2n}) = \frac{\pi n}{2n+1}$.*

Theorem M.2 ([HJ, Theorem 5.3])**.** *For all $n \geq 1$, $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^{2n+1}) = \frac{\pi n}{2n+1}$.*

The main findings of [R1], obtained using geometric constructions involving the spheres $\mathbb{S}^n$, are the following:

1. The GH distance between $\mathbb{S}^3$ and $\mathbb{S}^4$ is $\zeta_3 = \arccos\left(\frac{-1}{4}\right)$.
2. Different proofs of Theorems M.1 and M.2.

[13] In [LMS] they also consider the spheres $\mathbb{S}^0$ and $\mathbb{S}^\infty$, with appropriately defined metrics.

[R3]: GH distance and the fundamental group Another focal point of our project was the distance between $\mathbb{S}^1$ and simply connected geodesic spaces.

Definition M.3. A metric space (X, d_X) is *geodesic* if for all $x, y \in X$ there is an isometric imbedding $\iota : [0, d_X(x, y)] \to X$ with $\iota(0) = x$ and $\iota(d_X(x, y)) = y$.

A *metric tree* is a geodesic metric space (T, d_T) that contains no subspaces homeomorphic to $\mathbb{S}^1$; equivalently, for every $x, y \in T$ there is exactly one arc joining x and y.

For example, the intervals $[a, b]$ with their usual metric are metric trees, and spheres $\mathbb{S}^n$ with their intrinsic (also called 'geodesic') metric are geodesic.

A concrete case of [LMS, Theorem B] implies that $d_{\mathrm{GH}}(\mathbb{S}^1, \mathbb{S}^n) \geq \frac{\pi}{3}$ for all $n \geq 2$ (with equality for $n = 2, 3$). Similarly, in [Ka, Lemma 2.3] it was proved that for any interval $I \subseteq \mathbb{R}$ we have $d_{\mathrm{GH}}(I, \mathbb{S}^1) \geq \frac{\pi}{3}$, with equality being reached in intervals of length between $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

Via considerations related to persistent homology and the filling radius, in [LMO, Remark 9.19] it was deduced that for any compact geodesic metric space X one has $d_{\mathrm{GH}}(X, \mathbb{S}^1) \geq \frac{\pi}{6}$ and, taking into account the examples above, the following conjecture was formulated:

Conjecture M.4 ([LMO, Conjecture 4])**.** *For any compact, simply connected geodesic space X we have $d_{\mathrm{GH}}(X, \mathbb{S}^1) \geq \frac{\pi}{3}$.*

In [R3] the following result is obtained:

Theorem M.5. *Any simply connected geodesic space X satisfies $d_{\mathrm{GH}}(X, \mathbb{S}^1) \geq \frac{\pi}{4}$, and there is a simply connected geodesic space* E *with $d_{\mathrm{GH}}(\mathrm{E}, \mathbb{S}^1) = \frac{\pi}{4}$.*

The space E is in fact a metric tree (with the shape of the letter 'E'), and it is in fact the only metric tree of length at most $\frac{5\pi}{4}$ which is at distance $\frac{\pi}{4}$ of $\mathbb{S}^1$ (see [R3, Proposition 4.5]).

The lower bound for $d_{\mathrm{GH}}(X, \mathbb{S}^1)$ in Theorem M.5 is obtained as a corollary from a more general theorem relating GH distance and the fundamental group:

Theorem M.6 ([R3, Theorem 1.3])**.** *Let X, Y be geodesic spaces. Suppose there is a constant $D > 0$ such that $d_{\mathrm{GH}}(X, Y) < D$ and all loops of diameter $< 4D$ are nullhomotopic in X and Y. Then $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.*

Finally, the following is an elementary question which we have not been able to answer:

Question M.7. Is the space of compact metric trees modulo isometry, with the GH distance, a geodesic metric space? Equivalently, given two compact metric trees T, S, is there another metric tree X such that $d_{\mathrm{GH}}(X, T) = d_{\mathrm{GH}}(X, S) = \frac{1}{2} d_{\mathrm{GH}}(T, S)$?

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