What are smooth bump functions?

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1 Smooth bump functions.

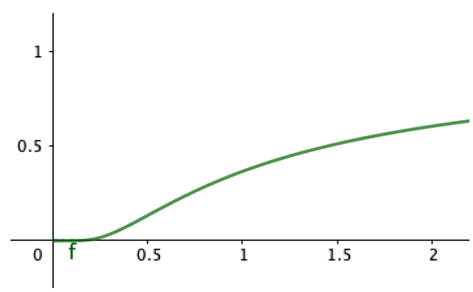
Introduction I will be talking about bump functions, and how we can use them to prove a few facts about smooth manifolds.

Definition 1.1. Let X be a topological space and $f : X \to \mathbb{R}$ be continuous. The *support* of f is $\operatorname{supp}(f) = \overline{\{x \in X; f(x) \neq 0\}}$. A smooth bump function in $\mathbb{R}^d$ is a C^∞ function $f : \mathbb{R}^d \to \mathbb{R}$ which has compact, nonempty support.

The first time one considers the concept of bump functions, it is not obvious that bump functions even exist. For example, there is no analytic bump function $f : \mathbb{R}^d \to \mathbb{R}$, as any analytic function which is 0 in a nonempty open set has to be 0 everywhere, by a connectedness argument.

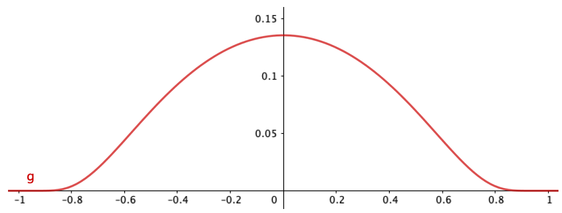
Construction of a C^∞ bump function. We start by noticing that there exist C^∞ functions $f : \mathbb{R} \to \mathbb{R}$ which are not identically 0, but such that its derivatives $f^{n)}(0)$ are 0 for all $n \geq 0$:

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

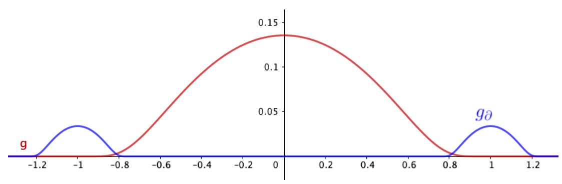


To check that $f^{n)}(0) = 0$ for all n one can use the fact that for all rational functions $r : \mathbb{R} \to \mathbb{R}$ we have $\lim_{x\to 0} e^{-1/x} \cdot r(x) = 0$ (equivalently, taking $y = -1/x$, for all rational functions we have $\lim_{y\to\infty} \frac{e^y}{r(y)} = \infty$).

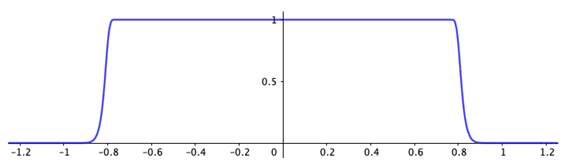
The function $g(x) = f(1+x) \cdot f(1-x)$ is a bump function, as it is supported in the interval $[0, 1]$.



Plateau bump functions. Bump functions can be made to have a nice property: they can have a plateau, by which we mean a nonempty open set in which the function is identically 1. To construct such a function, we can first take a bump function g_∂ supported near the boundary of $\mathrm{supp}(g)$; g_∂ can be obtained from a sum of two small copies of g.



And then, the 'quotient' $h = \frac{g}{g+g_\partial}$ (we define $h(x) = 0$ when $g(x) + g_\partial(x) = 0$) is a bump function with the plateau we want:



Bump functions in higher dimension. Once we have our bump function g in $\mathbb{R}$, we can easily create bump functions $\mathbb{R}^d \to \mathbb{R}$. For example, the function $(x, y) \mapsto g(x) \cdot g(y)$ will work, and so will $(x, y) \mapsto g(x^2 + y^2)$.

Smooth sums of bump functions. We can obtain smooth functions as infinite sums of bump functions. First a technical lemma:

Lemma 1.2. *Let $f_n : \mathbb{R}^d \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that $|\nabla f_n(x)| < \frac{1}{2^n}$ for all $x \in \mathbb{R}^d$ and $\sum_n f_n$ is defined. Then $f = \sum_n f_n$ is differentiable, and $f'(x) = \sum_n f_n'(x)$ for all $x \in \mathbb{R}^d$.*

Proof. We prove that f is differentiable at 0. Let $v = \sum_n \nabla f_n(0)$; we want to prove that $\nabla f(0) = v$. That is, for every $\varepsilon > 0$ we want to find δ such that, for all x with $|x| < \delta$, we have $|f(x) - v \cdot x| < \varepsilon|x|$. And indeed, let N be such that $2^{-N} < \varepsilon/2$ and $\left|v - \sum_{n=1}^N f'(0)\right| < \varepsilon/2$, so that $\sum_{n=N+1}^\infty f_n$ is $\varepsilon/2$-Lipschitz, and let $\delta > 0$ be such that $\left|\sum_{n=1}^N f(x) - v \cdot x\right| < \varepsilon|x|/2$ for all x with $|x| < \delta$.

Then, for all $x \in B(0,\delta)$,

$$|f(x) - v\cdot x| \le \left|\sum_{n=1}^{N} f(x) - v\cdot x\right| + \left|\sum_{n=N+1}^{\infty} f_n(x)\right| < \varepsilon|x|/2 + \varepsilon|x|/2 = \varepsilon|x|. \quad \square$$

Corollary 1.3. *Let $\phi_n : \mathbb{R}^d \to \mathbb{R}$ be a bump*[1] *function for all n. If a sequence (λ_n) of positive real numbers decreases to 0 fast enough, then $f = \sum_n \lambda_n \phi_n$ is smooth.*

Proof. Choose the constants λ_n so that all the partial derivatives of order $\le n$ of ϕ are $\le \frac{1}{10^n}$ for all n, and use Lemma 1.2 to prove by induction that f is of type C^n for all n. $\square$

Some constructions using bump functions Here are a few existence results for C^∞ functions that are false for analytic functions.

Proposition 1.4. *Any closed set $C \subseteq \mathbb{R}^d$ is the zero set of some smooth function $f : \mathbb{R}^d \to [0,\infty)$.*

Proof. Express $\mathbb{R}^d \setminus C = \cup_n B_n$ as a countable union of balls, and let ϕ_n be a bump function which is positive exactly in B_n and 0 elsewhere. We can just let $f = \sum_n \lambda_n \phi_n$, for some sequence (λ_n) which decreases fast enough. $\square$

Proposition 1.5 (Smooth Urisohn lemma)**.** *For any disjoint closed sets $C_0, C_1 \subseteq \mathbb{R}^d$, there is a function $f : \mathbb{R}^d \to [0,1]$ such that $C_0 = f^{-1}(\{0\})$ and $C_1 = f^{-1}(\{1\})$.*

Proof. Let $g_0, g_1 : \mathbb{R}^d \to [0,\infty)$ be smooth functions with $g_0^{-1}(\{0\}) = C_1$ and $g_1^{-1}(\{0\}) = C_0$. Then the function $h(x) = \frac{g_1}{g_0+g_1}(x)$ works.[2] $\square$

Extending and combining smooth functions We can use Proposition 1.5 to construct extensions of smooth functions. For a closed set $C \subseteq \mathbb{R}^d$, say a $f : C \to \mathbb{R}$ is smooth if it can be extended to a smooth function in an open set containing C[3].

Proposition 1.6. *If $C \subseteq \mathbb{R}^d$ is closed and $f : C \to \mathbb{R}$ is smooth, then f admits an extension $F : \mathbb{R}^d \to \mathbb{R}$.*

Proof. Let $\overline{f}$ be an extension of f to an open set $O \supseteq C$, let $C_0 = \mathbb{R}^d \setminus O$ and let g be a smooth function which is 1 in the set C and 0 in the set $\{x \in \mathbb{R}^d; d(x, C_0) \le d(x, C)\}$. Then the function $h = \overline{f} \cdot g$ (extending by $h(x) = 0$ if $x \in C_0$) works. $\square$

[1] This works for any sequence of smooth functions, not only bump functions, using that smooth functions have their derivative bounded in compacts sets.

[2] This construction, simpler than the one I had in mind, was provided by Prof. Neil Falkner during the talk.

[3] We could give an a priori weaker, local definition of differentiability; one can check that they are equivalent using partitions of unity.

In particular, we can glue smooth functions:

Corollary 1.7. *For any disjoint closed sets $C_1, \ldots, C_n \subseteq \mathbb{R}^d$ and any smooth functions $f_i : C_i \to \mathbb{R}$, there is a smooth function $f : \mathbb{R}^d \to \mathbb{R}$ such that $f(x) = f_i(x)$ for all $i = 1, \ldots, n$ and $x \in C_i$.*

Proof. Apply Proposition 1.6 to the (smooth) function $g : \cup_i C_i \to \mathbb{R}$ given by $g(x) = f_i(x)$ for $x \in C_i$. □

Counterexamples using bump functions We first recall a famous calculus counterexample: the function $f(x, y) = \frac{xy}{x^2+y^2}$ is not differentiable at 0, but all its directional derivatives exist. Equivalently, f is smooth when restricted to any line, but is not continuous in $\mathbb{R}^2$.

By a curve $\gamma : \mathbb{R} \to \mathbb{R}^d$ being smooth, we mean that it is C^∞ and that $\gamma'(t) \neq 0$ for all $t \in [0, 1]$.

Proposition 1.8. *There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not continuous, but such that for all smooth curves $\gamma : \mathbb{R} \to \mathbb{R}^2$ the function $f(\gamma(t))$ is smooth.*

For a proof see this link; the key fact used in the proof is that there is a sequence of balls $B_n \subseteq \mathbb{R}^2$ convergent to 0 and such that the image of any smooth curve $\gamma : [0, 1] \to \mathbb{R}^2$ intersects only finitely many of them.

Proposition 1.9. *There is a function $f : \mathbb{R}^3 \to \mathbb{R}$ which is not continuous, but whose restriction to every plane is smooth.*

For a proof see this link. Again, the key fact is that there is a sequence of balls $B_n \subseteq \mathbb{R}^3$ convergent to 0, but such that each plane can intersect at most 3 of them.

Mollifiers. We can use bump functions supported near 0 to create 'explicit' C^∞ approximations of continuous functions f in compact sets.

Let $g : \mathbb{R}^d \to [0, \infty]$ be a bump function with $\int_{\mathbb{R}^d} g \, d\mu = 1$ and supported in the ball $B(0, \delta)$, for some small $\delta > 0$. We can consider g as the density function of a random variable V, so that V takes values in $\mathbb{R}^d$, and has norm $< \delta$ with probability 1.

Now, given a continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we consider the function

$$g(x) = \mathbb{E}(f(x + V)) = \int_{\mathbb{R}^d} f(x + v)g(v)d\mu(v).$$

Even if f was an arbitrary continuous function, the function g is smooth!

Definition 1.10 (Convolution)**.** Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be continuous, with g being compactly supported. We define $f * g : \mathbb{R}^d \to \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x)g(t - x)d\mu(x)$$

Fact 1.11. *If f, g are as above and g is smooth, then $\frac{\partial}{\partial x_i}(f*g)(x) = \left(f * \frac{\partial g}{\partial x_i}\right)(x)$. So the convolution of a continuous function and a bump function is smooth.*

As we make δ go to 0, the function g will converge to f, uniformly in compact sets: indeed, given $\varepsilon > 0$ and a compact set $K \subseteq \mathbb{R}^d$, choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x \in K, y \in \mathbb{R}^d$ such that $|x - y| < \varepsilon$. Then, if g is supported in $B(0, \delta)$, for all $x \in K$ we necessarily have $|f(x) - g(x)| < \delta$. This is because the variable $x + V$ takes values inside $B(x, \delta)$ with probability 1, so $f(x + V)$ is inside $B(f(x), \varepsilon)$ with probability 1, so $\mathbb{E}(f(x + V)) \in B(f(x), \varepsilon)$.

This can be seen as a 'more explicit' alternative to Weierstrass' approximation theorem; however, one can prove much stronger results, like this stronger version of the Weierstrass approximation theorem for analytic functions, which was proved by Carleman in dimension 1 and by Scheinberg in higher dimension.

Proposition 1.12 ([Sch])**.** *If $\eta, \varphi : \mathbb{R}^d \to \mathbb{R}$ are continuous and $\eta > 0$, then there exists an entire function $\Psi : \mathbb{C}^d \to \mathbb{C}$ such that for all real x, $|\Psi(x) - \varphi(x)| < \eta(x)$.*

References

[Sch] Scheinberg. *Uniform approximation by entire functions* J. Analyse Math. 29 (1976), 16–18.