What are Cantor Spaces?

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The Cantor Space, $2^{\mathbb{N}}$

$$2^{\mathbb{N}} := \{0,1\}^{\mathbb{N}} = \{(a_n)_{n=1}^{\infty}; a_n \in \{0,1\} \forall n\}.$$

We give it the product topology, so it is compact.

Basic open sets: for any $w = w_0 w_1 \dots w_k$ finite word,

$$O_w := \{(a_n)_n; a_1 \dots a_k = w_1 \dots w_k\}.$$

The sets O_w are actually clopen.

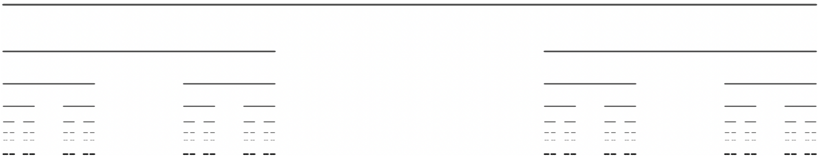
$$\begin{aligned} 2^{\mathbb{N}} &= O_0 \cup O_1 = O_{00} \cup O_{01} \cup O_{10} \cup O_{11} \\ &= O_{000} \cup O_{001} \cup O_{010} \cup O_{011} \cup O_{100} \cup O_{101} \cup O_{110} \cup O_{111} = \dots \end{aligned}$$

Points of $2^{\mathbb{N}}$ can be expressed in terms of this topology:

$$2^{\mathbb{N}} \longleftrightarrow \left\{ \begin{array}{c} \text{Decreasing sequences } O_{w_1} \supseteq O_{w_2} \subseteq O_{w_3} \supseteq \dots \\ \text{With } w_n \text{ having length } n. \end{array} \right\}$$

The Ternary Cantor Set, $\mathcal{C}$

$$\mathcal{C} := \left\{ \sum_{n=1}^{\infty} (2a_n)3^{-n};\, a_n \in \{0,1\} \forall n \right\} \subseteq [0,1].$$



The ternary Cantor set also has a basis of clopen sets $\{A_w; w \text{ finite word}\}$ (draw picture).
We also have a bijection

$$\mathcal{C} \longleftrightarrow \left\{ \begin{array}{c} \text{Decreasing sequences } A_{w_1} \supseteq A_{w_2} \subseteq A_{w_3} \supseteq \dots \\ \text{With } w_n \text{ having length } n. \end{array} \right\}$$

$2^{\mathbb{N}}$ and $\mathcal{C}$ are homeomorphic via $\bigcap_n A_{w_n} \mapsto \bigcap_n O_{w_n}$.

Brouwer's theorem (1910)

Theorem (Brouwer,1910)

Let $X \neq \varnothing$ be any compact, totally disconnected metric space without isolated points. Then X is homeomorphic to $2^{\mathbb{N}}$.

Sketch of proof.

As X is disconnected, we can divide it into two nonempty clopen subspaces A_0, A_1.
A_0, A_1 also are compact, totally disconnected without isolated points, so we can subdivide them into $A_{00}, A_{01}, A_{10}, A_{11}$.
Repeating this process infinitely, we obtain clopens A_w for each w finite $\{0,1\}$-word.
One can construct in this manner a family $(A_w)_w$ such that for each sequence $a := (a_n)_n \in 2^{\mathbb{N}}$, the intersection of the decreasing sequence $A_{a_1} \supseteq A_{a_1 a_2} \supseteq A_{a_1 a_2 a_3} \dots$ is one point p_a (see next slide for details if interested).
The map $p_a \mapsto a$ is then a homeomorphism from $2^{\mathbb{N}}$ to X, because it is a continuous bijection from compact to Hausdorff. $\square$

Construction of the family $\{A_w\}_w$.

Here are steps one can follow to prove that the sets A_w from the previous proof can be constructed in such a way that the decreasing sequences $\bigcap_n A_{w_n}$ of clopens have intersection just one point:

1. If X is a compact, Hausdorff space, then connected components and quasi-components coincide in X.
2. A space X satisfying the hypotheses of Brouwer's theorem has a basis formed by clopen sets.
3. A space X satisfying the hypotheses of Brouwer's theorem has a countable basis $(C_n)_n$ formed by clopen sets.
4. Construct the sets A_w such that, if w has length n, then either $A_w \subseteq C_n$ or $A_w \subseteq X \setminus C_n$.

Examples of Cantor Spaces

Let $X \neq \varnothing$ be any compact, totally disconnected metric space without isolated points. Then X is homeomorphic to $2^{\mathbb{N}}$.

We say a topological space X is a *Cantor Space* if it is homeomorphic to $2^{\mathbb{N}}$. For example,

- $2^{\mathbb{N}}, \mathcal{C}$.
- Countable products of finite discrete spaces* (e.g. $3^{\mathbb{N}}$).
- $\mathcal{C}^2 \subseteq [0,1]^2$.
- Countable products of Cantor spaces (e.g. $\mathcal{C}^{\mathbb{N}}$).
- $2^{\mathbb{N}} \times X$, where X is any compact, totally disconnected metric space.

Continuous images of $2^{\mathbb{N}}$

Theorem (Hausdorff-Alexandroff,1927)

Let $X \neq \varnothing$ be a compact metric space. Then there is a surjective map $f : 2^{\mathbb{N}} \to X$.

Proof.

For $n \in \mathbb{N}$ let F_n be a finite, $\frac{1}{2^n}$-dense subspace of X.
Let $Y := \prod_n F_n = \{(p_n)_n; p_n \in F_n \text{ for all } n\}$. Y is compact, totally disconnected.
Let $Z = \{(p_n)_n \in Y; d(p_n, p_{n+1}) < \frac{1}{2^{n-1}} \forall n\}$. Z is closed in Y, so it is compact, totally disconnected.
The map $Z \to X; (p_n)_n \mapsto \lim_n p_n$ is surjective.
The projection $2^{\mathbb{N}} \times Z \to Z$ is also surjective, and $2^{\mathbb{N}} \times Z$ is a Cantor space, so we are done. $\square$

Space filling curves

Theorem

Let $A \subseteq \mathbb{R}^n$ compact, convex set. Then there is a surjective curve $c : [0,1] \to A$.

Proof sketch.

As A is compact, there is a surjective map $f : \mathcal{C} \to A$. We can extend it to a map $\overline{f} : [0,1] \to A$ by interpolating linearly in the connected components of $[0,1] \setminus \mathcal{C}$. □

What topological spaces X have a surjective curve $[0,1] \to X$?

Definition

A *Peano continuum* is a compact, connected, locally connected metrizable topological space $X \neq \varnothing$.

Theorem (Hahn-Mazurkiewicz, 1920)

A Hausdorff topological space is a continuous image of $[0,1]$ *iff it is a Peano continuum. (Also, any locally compact, connected, locally connected metrizable topological space $X \neq \varnothing$ is an image of $\mathbb{R}$.)*

For more nice consequences of the Hausdorff-Alexandroff theorem, see Yoav Benyamini's 'Applications of the universal surjectivity of the Cantor Set'.

Subspaces of $2^{\mathbb{N}}$

Theorem

A Hausdorff space X is homeomorphic to some subspace of $2^{\mathbb{N}}$ iff it has a countable base formed by clopen sets.

Sketch of proof.

Let $(C_n)_n$ be the basis of X formed by clopen sets. To any point $x \in X$ we can associate a $\{0,1\}$-sequence a_x given by $a_x(n) = 0$ if $x \in C_n$ and $a_x(n) = 1$ if $x \notin C_n$.
The map $x \mapsto a_x$ is an imbedding of X into $2^{\mathbb{N}}$. $\square$

Also:

- Any nonempty clopen subspace of $2^{\mathbb{N}}$ is a Cantor space.
- Any nonempty open subspace of $2^{\mathbb{N}}$ is either a Cantor space or homeomorphic to $2^{\mathbb{N}} \setminus \{*\}$.

Homogeneity properties of Cantor Spaces

- $2^{\mathbb{N}}$ is a topological group, so it is homogeneous.
- $2^{\mathbb{N}}$ is n-homogeneous for all n.

A separable topological space X is countable dense homogeneous if for any two dense countable dense subsets E, D of X there is a homeomorphism $f : X \to X$ with $f(D) = E$.

Theorem

$2^{\mathbb{N}}$ is countable dense homogeneous. (Proof similar to Brouwer theorem.)

Stronger statement: $(E_n)_n$ and $(D_n)_n$ sequences of pairwise disjoint countable dense subsets of $2^{\mathbb{N}}$, then there is a homeo $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ sending E_n to D_n for all n.

Corollary (Sierpinski Theorem, 1920)

Any countable metric space X without isolated points is homeomorphic to $\mathbb{Q}$.

So $\mathbb{Q} \cong \mathbb{Q} \cap [0, 1] \cong \mathbb{Q}^n \cong \mathbb{A}$.

The Baire space, $\mathbb{N}^\mathbb{N}$

It is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Theorem (Hausdorff, 1937)

If a topological space X is completely metrizable, not locally compact at any point and has a countable basis of clopen sets, then it is homeomorphic to $\mathbb{N}^\mathbb{N}$.

(Proof similar to Brouwer's theorem)

- A Hausdorff space X is homeomorphic to some subspace of $\mathbb{N}^\mathbb{N}$ iff it has a countable base formed by clopen sets.
- $\mathbb{N}^\mathbb{N}$ is homeomorphic to any open nonempty subset of itself.
- $\mathbb{N}^\mathbb{N}$ is also countable dense homogeneous.
- Nice homeomorphism $\mathbb{N}^\mathbb{N} \to (1, \infty) \setminus \mathbb{Q}$:

$$(a_n)_n \mapsto a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$