The lemniscate: from Bernoulli to Gauss

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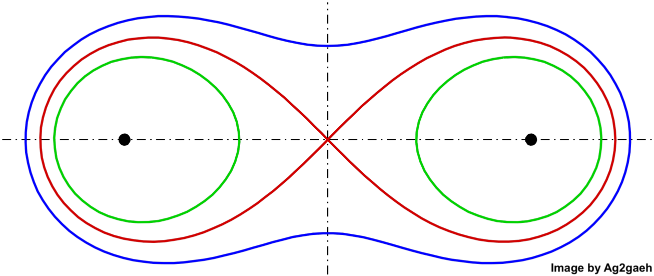
Cassini ovals

A Cassini oval is the set of points $R \in \mathbb{R}^2$ such that

$$d(P,R) \cdot d(Q,R) = \lambda,$$

for some given $P, Q \in \mathbb{R}^2$ and $\lambda \geq 0$. P, Q are the *foci* of the oval.



If $P = (-a, 0)$ and $Q = (a, 0)$, then the Cassini oval has equation

$$((x+a)^2 + y^2) \cdot ((x-a)^2 + y^2) = \lambda^2.$$

Expanding the above as a polynomial on a, we obtain

$$a^4 - 2a^2(x^2 - y^2) + (x^2 + y^2)^2 = \lambda^2.$$

Cassini Ovals

The shape of a Cassini oval is determined by a unique parameter, the excentricity, given in the example above by $e = \frac{\sqrt{\lambda}}{a}$. The oval is connected iff $e \geq 1$.

These ovals were first investigated by Giovanni Cassini in 1680 while studying the relative motions of the Earth and the Sun. He mistakenly believed that the movement of the sun, with the Earth as a frame of reference, had the shape of a Cassini oval with the Earth in one focus.

Cassini Ovals in tori

The torus T, viewed as a "donut" shape in $\mathbb{R}^3$ centered around the z-axis, is given by

$$\left(\sqrt{x^2+y^2}-R\right)^2+z^2=r^2.$$

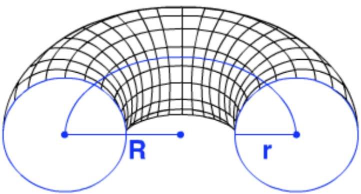


Image by Jenny Eather

The intersection of T with the plane $y=r$ is a Cassini oval. Indeed, the resulting equation is

$$\begin{aligned}\left(\sqrt{x^2+r^2}-R\right)^2+z^2&=r^2;\\ x^2+r^2+R^2-2R\sqrt{x^2+r^2}+z^2&=r^2;\\ (x^2+z^2+R^2)^2&=4R^2(x^2+r^2)\\ R^4-2R^2(x^2-z^2)+(x^2+z^2)^2&=4R^2r^2.\end{aligned}$$

That is, a Cassini oval with foci $(\pm R, 0)$ and product of distances $2Rr$ (see geogebra file due to Ignacio Larrosa).

Lemniscate of Bernoulli

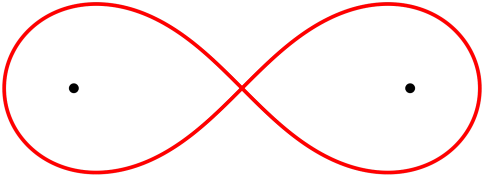
Several curves with shape similar to the ∞ symbol are known as lemniscates.

A lemniscate of Bernoulli is a Cassini oval with excentricity 1. The lemniscate with foci $(\pm a, 0)$ has equation

$$a^4 - 2a^2(x^2 - y^2) + (x^2 + y^2)^2 = a^4,$$

or equivalently,

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$



The equation in polar coordinates is especially simple, $r^2 = 2a^2 \cos(2\theta)$.

Lemniscate of Bernoulli

The Lemniscate of Bernoulli was first described by Bernoulli in 1694, in an article in *Acta Eruditorum*.

This raises a natural question: which Bernoulli?

Jakob described the curve in an article which appeared in September, and Johann in October. They seemingly discovered the curve independently, as both of them were trying to solve the same problem with the same tools.

What they were trying to solve was the paracentric isochrone problem (animation), posed by Leibniz in 1689.

It turns out the paracentric isochrone can be parameterized explicitly using the arc-length function of the lemniscate, an elliptic integral.

The lemniscate and inversion

Circle inversion (around the unit circle) is the transformation of the Euclidean plane given in polar coordinates by

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2;$$
$$(x, y) \mapsto \frac{(x, y)}{x^2 + y^2},$$

or in polar coordinates, $(r, \theta) \mapsto \left(\frac{1}{r}, \theta\right)$.
A nice property of circle inversion is that it preserves lines/circles.
The inverse of a hyperbola whose asymptotes form 45° is a lemniscate.
Inversion can be useful when solving some euclidean geometry problems; as far as we know it was discovered independently by several people (Steiner, Quetelet and others) at the beginning of the 19^{th}. century.

An aside: lengths of (non-straight) curves

Since ancient times, people who studied curves were aware that non-straight curves have a definite length. For example, the Babylonians assumed that the circumference of a circle was three times its diameter.

In his Elements, Euclid used Eudoxus' method of exhaustion to prove that 'circles are to one another as the squares of their radii'.

Archimedes, in his *Sphere and Cylinder*, proved that $3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$ by estimating the length of the unit circumference. He made the following assumption:

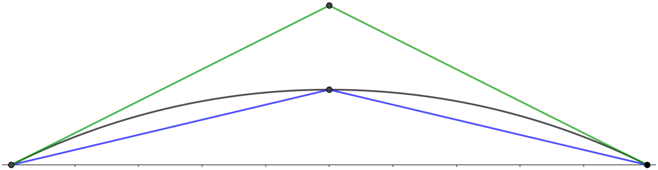
Lengths of curves

A curve is concave in one direction if every line which connects two of its points has all points of the curve lying on one side of the line or on the line itself and none on the other side. It is then assumed (as axioms):

(A) Of all lines connecting two points the straight line is the shortest.

(B) Of all lines in the plane having the same extremities two are unequal when both are concave in the same direction, and that one of them included between the other and the straight line connecting the two points is the lesser.

(C) The length of a curve is a magnitude under the definition of Euclid X



This allowed him to find upper/lower bounds for the length of a circle, using polygons inscribed and circumscribed around the circle. He also claimed that for all $\lambda > 1$ there are polygons inscribed and circumscribed in the circle so that the ratio of their lengths is $< \lambda$.

Lengths of curves

But Archimedes was not done yet. In his treatise *On Spirals*, he developed another, completely different method to find lengths of curves. Quoting [Coo],

> *There a curve is looked on kinematically; it is traced by a point which moves by two different impetuses, and the length depends upon the relative strength of the two.*

This is similar to the modern definition of length of a smooth curve $\gamma(t)$, where $a \leq t \leq b$; the length is given by

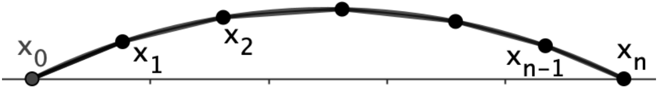
$$\int_a^b |\gamma'(t)| dt \stackrel{\mathbb{R}^2}{=} \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

A general definition of length of a curve

Nowadays we have a more general definition of length of a curve $\gamma : [a, b] \to X$, where (X, d) is an arbitrary metric space:

$$\sup_{a=x_0<x_1<\cdots<x_n=b} d(\gamma(x_0), \gamma(x_1)) + \cdots + d(\gamma(x_{n-1}), \gamma(x_n)).$$

This is reminiscent of Archimedes' method to approximate the length of a curve by an inscribed polygon:



The length of the Lemniscate

G.C. di Fagnano (1682-1766) was known for his investigations about the arc lengths of certain curves, and he seemed to give much importance to his work on the lemniscate. He had a lemniscate engraved in the title page of his *Produzioni Matematiche.*

PRODUZIONI MATEMATICHE

DEL CONTE GIULIO CARLO
DI FAGNANO,
MARCHESE DE' TOSCHI,
E DI SANT' ONORIO
NOBILE ROMANO, E PATRIZIO SENOGAGLIESE

ALLA SANTITA' DI N. S.

BENEDETTO XIV.
PONTEFICE MASSIMO.

TOMO PRIMO.

MULTIFARIAM DIVISA ATQUE DIMENSA

DEO VERITATIS GLORIA

Jos. Gavelli del. et sc.

IN PESARO

L' ANNO DEL GIUBBILEO M. DCC. L.
NELLA STAMPERIA GAVELLIANA
CON LICENZA DE' SUPERIORI.

The length of the Lemniscate. Elliptic integrals

It was Fagnano and Euler who first studied elliptic integrals during the 18^{th} century.

'Elliptic integral' is an umbrella term for some functions of the form

$$f(x) = \int_c^x R\left(t, \sqrt{P(t)}\right) dt,$$

where $R \in \mathbb{R}[x, y]$ is a rational function and P is a polynomial of degree 3 or 4 without repeated roots. They are called 'elliptic' because one can use them to compute the length of an ellipse.

But it seems the first time elliptic integrals were used was in order to compute the length of the lemniscate, which we will now express as a simple integral.

The length of the lemniscate

One can check that, for $r \in [0,1]$, the points of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ at distance r of the origin are given by

$$\left(\pm\sqrt{\frac{1}{2}r^2(1+r^2)}, \sqrt{\pm\frac{1}{2}r^2(1-r^2)} \right), r \in [0,1].$$

Therefore, we can parameterize a quarter of the lemniscate as

$$(x(r), y(r)) = \left(\sqrt{\frac{1}{2}r^2(1+r^2)}, \sqrt{\frac{1}{2}r^2(1-r^2)} \right), r \in [0,1].$$

So the length of a lemniscate arc from the origin to $(x(r), y(r))$ is given by

$$\int_0^r \sqrt{x'(t)^2 + y'(t)^2} dt = \cdots = \int_0^r \frac{1}{\sqrt{1-t^4}} dt.$$

Lemniscatic trigonometry

The constant ϖ is used to denote the length of a lemniscate of diameter 1, same as π is the length of a circle of diameter 1.

$$\varpi = 2\int_0^1 \frac{1}{\sqrt{1-t^4}}dt \approx 2.62205755.$$

We can also define lemniscatic sine and cosine functions with nice properties. Using that $\arcsin'(x) = -\arccos'(x) = \frac{1}{\sqrt{1-x^2}}$, we can deduce that, for $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$,

$$x = \int_0^{\sin(x)} \frac{1}{\sqrt{1-t^2}}dt = \int_{\cos(x)}^1 \frac{1}{\sqrt{1-t^2}}dt.$$

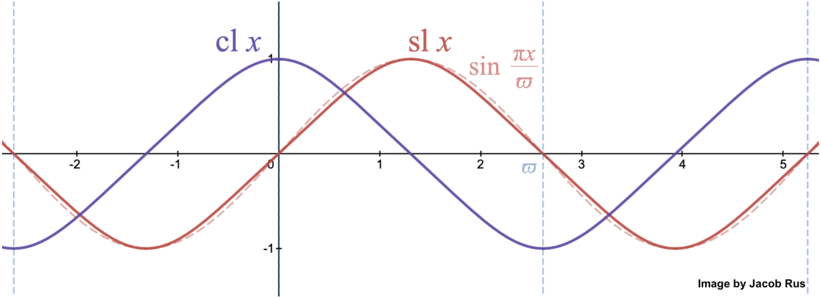
We define the lemniscatic sine and cosine, $\mathrm{sl}, \mathrm{cl} : \left(\frac{-\varpi}{2}, \frac{\varpi}{2}\right) \to [0,1]$, by

$$x = \int_0^{\mathrm{sl}(x)} \frac{1}{\sqrt{1-t^4}}dt = \int_{\mathrm{cl}(x)}^1 \frac{1}{\sqrt{1-t^4}}dt.$$

sl, cl can be extended differentiably to almost all $\mathbb{C}$.

Lemniscatic trigonometry

Here is how sl, cl look.



sl, cl don't seem to have such a nice geometric definition as $\sin, \cos$. The lemniscatic elliptic functions were first studied (but not published) by Gauss in 1797; they are a concrete class of Jacobi elliptic functions, introduced by Jacobi in 1829.

Lemniscatic trigonometry

sl, cl have some similar properties to $\sin, \cos$.

Trigonometric	**Lemniscatic**
$\sin^2(x) + \cos^2(x) = 1$	$(1 + \mathrm{sl}^2(x)) + (1 + \mathrm{cl}^2(x)) = 2$
$\sin(-x) = -\sin(x)$	$\mathrm{sl}(-x) = -\mathrm{sl}(x)$
$\cos(-x) = \cos(x)$	$\mathrm{cl}(-x) = \mathrm{cl}(x)$
$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$	$\mathrm{sl}\left(x + \frac{\varpi}{2}\right) = \mathrm{cl}(x)$
	$\mathrm{sl}(iz) = i\,\mathrm{sl}(z)$, $\mathrm{cl}(iz) = \frac{1}{\mathrm{cl}(z)}$.
$\sin(x)$ is holomorphic	$\mathrm{sl}(x)$ is meromorphic, poles at $\varpi\left(a + \frac{1}{2}\right) + i\varpi\left(b - \frac{1}{2}\right)$, $a, b \in \mathbb{Z}$
Periodic, periods $2\pi n$, $n \in \mathbb{Z}$	Doubly periodic, periods $\varpi(a + bi)$, $a, b \in \mathbb{Z}$, $a + b$ even

And many more, $\mathrm{sl}(x + y), \mathrm{sl}'(x), \cdots$ have reasonable expressions.

Gauss and ϖ. The arithmetic-geometric mean

Let $x < y$ be positive numbers. Recall that, if we define the arithmetic and geometric mean of x, y,

$$A_1 = \frac{x+y}{2}, G_1 = \sqrt{xy},$$

then $G_1 < A_1$, because $(\sqrt{x} - \sqrt{y})^2 > 0$.Thus,

$$x < G_1 < A_1 < y.$$

Moreover, $A_1 - G_1 < y - A_1 = \frac{y-x}{2}$. This implies that, if we define for $n \geq 2$

$$A_n = \frac{A_{n-1} + G_{n-1}}{2}, G_n = \sqrt{A_{n-1}G_{n-1}},$$

then $G_n < G_{n+1} < A_{n+1} < A_n$, and $A_{n+1} - G_{n+1} < \frac{A_n - G_n}{2}$. So the decreasing sequence $(A_n)_{n\in\mathbb{N}}$ and the increasing sequence $(G_n)_{n\in\mathbb{N}}$ converge to a single value, which we denote

$$\mathrm{AGM}(x, y).$$

Gauss and ϖ. The arithmetic-geometric mean

The AGM was introduced by Lagrange in 1785, and later independently rediscovered and studied more in detail by Gauss. Both of them encountered the AGM while trying to compute elliptic integrals.
Gauss computed a few decimal places of ϖ and $\mathrm{AGM}\left(1, \sqrt{2}\right)$, and conjectured:

> We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is π/ϖ to 11 places; the proof of this fact will certainly open up a new field of analysis.
>
> May 30, 1799

He later proved the more general fact that, if $a \geq b > 0$, then

$$\mathrm{AGM}(a, b) \cdot \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{a^2 \cos^2(\theta) + b^2 \sin^2(\theta)}} d\theta = \frac{\pi}{2},$$

where the integral above can be seen as an elliptic integral using changes of variables.

- Wikipedia - Cassini Oval, Lemniscate, Lemniscate elliptic functions, Jacobi elliptic functions.
- C. Seligman. Cassini Measures the Orbit of the Sun.
- Popescu-Pampu. *What is the Genus?* (Chapter 6: Fagnano and the Lemniscate.) Springer, 2016.
- Bos. *The lemniscate of Bernoulli.* Translation of a paper read to the Congress of Dutch Mathematicians at Delft on April 6th, 1970.
- Coolidge. *The lengths of curves.* The American Mathematical Monthly, Vol. 60, No. 2 (Feb., 1953), pp. 89-93.
- Barnett *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve.*
- Cox. *The arithmetic-geometric mean of Gauss.* Pi: A source book. New York, NY: Springer New York, 1997. 481-536.