Early history of convex sets

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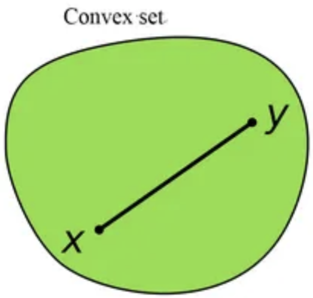
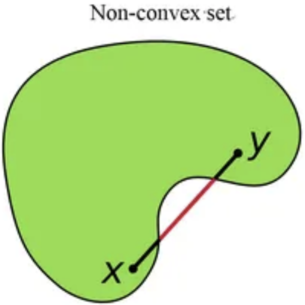
Convex sets

We first review some basics about convex sets.

Definition

A subset C of $\mathbb{R}^d$ is said to be convex if, for all points $x, y \in C$, the line segment $[x, y]$ is contained in C.





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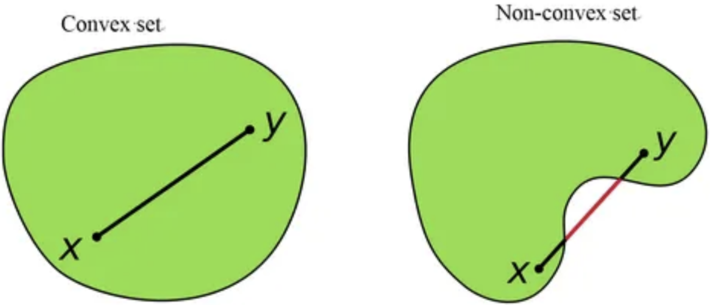
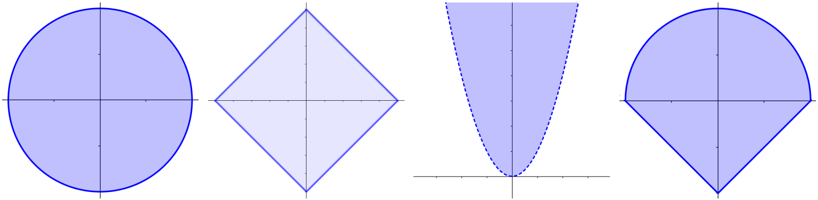


The intersection of any family of convex sets is convex.

Convex sets



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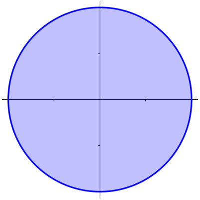

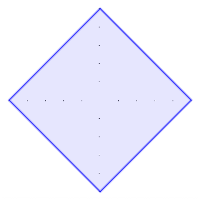

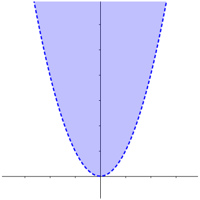

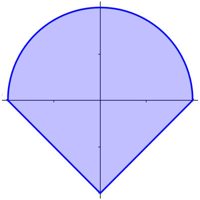

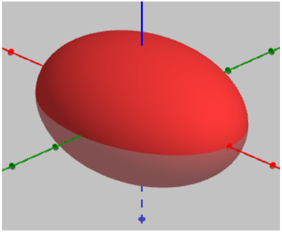

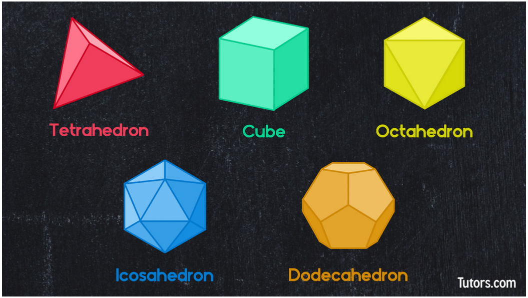
Tetrahedron
Cube
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Convex sets and convex functions

If $C \subseteq \mathbb{R}^d$ is convex, we say $f : C \to \mathbb{R}$ is convex if, for all $p, q \in C$ and $\lambda \in [0, 1]$,

$$f(\lambda p + (1 - \lambda)q) \leq \lambda f(p) + (1 - \lambda) f(q).$$

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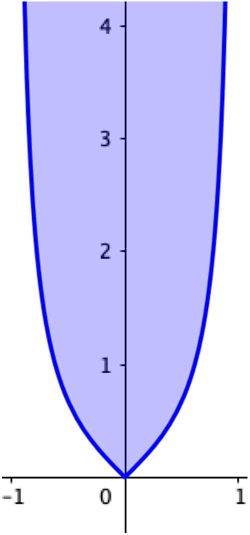
f is convex iff the set above the graph of the function,

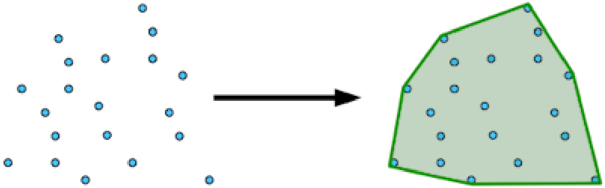
$$\{(x,y) \in \mathbb{R}; y \geq f(x)\},$$

is convex.



Convex hull, convex polyhedra

If $X \subseteq \mathbb{R}^d$, $\mathrm{Conv}(X)$ is the smallest convex set containing X (or, the intersection of all convex sets containing X).



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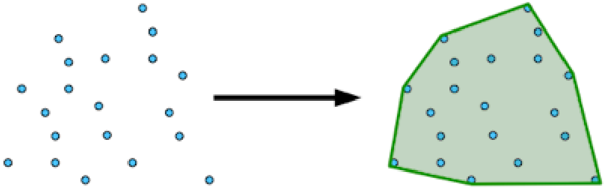

Conv(X) is also the set of affine combinations of points of X:

$$\mathrm{Conv}(X) = \left\{ \sum_{i=1}^{N} a_i x_i ; N \in \mathbb{N}, a_i \geq 0, \sum_i a_i = 1, x_i \in X \right\}$$
$$= \left\{ \sum_{i=1}^{d+1} a_i x_i ; a_i \geq 0, \sum_i a_i = 1, x_i \in X \right\}$$

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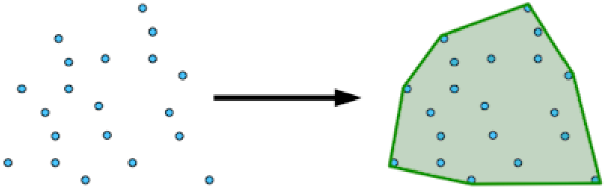

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A *convex polyhedron* is the convex hull of a finite subset of $\mathbb{R}^d$.

Some more properties of convex sets

- All convex sets in $\mathbb{R}^d$ have a well-defined dimension, i.e. if $C \subseteq \mathbb{R}^d$ is convex then there is an affine subspace X of $\mathbb{R}^d$ such that $C \subseteq X$ and C has nonempty interior in X.

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  - The boundary ∂C of a convex set $C \subseteq \mathbb{R}^2$ is a differentiable curve at all points except countably many.
  - More generally, the boundary ∂C of a convex set $C \subseteq \mathbb{R}^d$ is differentiable at all points except some set of Hausdorff dimension $d - 2$.

The greeks

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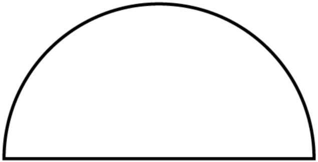
Archimedes continued the study of convex polyhedra, describing all 13 Archimedean solids, and in his treatise *On the Sphere and the Cylinder* he gave definitions of convexity for curves and surfaces (these curves/surfaces being the boundary of a convex set).

2. I apply the term **concave in the same direction** to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side.



4. I apply the term **concave in the same direction** to surfaces such that, if any two points on them are taken, the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it, but none on the other side.

$18^{\text{th}}, 19^{\text{th}}$ centuries

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Brunn called convex sets 'egg bodies', and their surfaces 'egg surfaces'.

Minkowski

In 1896, Minkowski (1864-1909) presented his book *Geometrie der Zahlen*, The Geometry of Numbers, where he studied convex sets in detail. His book gave rise to an area of number theory, called geometric number theory.

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The minimum problem

Given a positive definite quadratic form f in $\mathbb{R}^d$, that is

$$f : \mathbb{R}^d \to \mathbb{R}; f(x_1, \dots, x_d) = \sum_{i,j \leq d} a_{ij} x_i x_j,$$

for some real matrix $M = (a_{ij})_{i,j=1,\dots,d}$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$, what is the minimum value of $f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^d \setminus \{0\}$?

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In a letter to Jacobi, published in Crelle's Journal in 1850, Hermite had found that this minimum value is at most

$$\left(\frac{4}{3}\right)^{\frac{1}{2}(d-1)} \det(M)^{1/d}.$$

Minkowski improved this result considerably; using geometric reasonings (which we will not explain in detail), he was able to prove that the minimum is at most

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He then applied to this ball, B, the following nice result.

Convex body theorem

If a convex, symmetric subset of $\mathbb{R}^d$ has area $> 2^d$, then it contains some point in $\mathbb{Z}^d \setminus \{0\}$.

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(Cue rotating ellipses of area 4)

Convex sets and norms

Recall that a norm in $\mathbb{R}^d$ is a map $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ which to each $x \in \mathbb{R}^d$ associates a norm $\|x\|$, satisfying:

- $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \mathbb{R}^d$ (triangle inequality).
- $\|ax\| = |a| \cdot \|x\|$ for $a \in \mathbb{R}, x \in \mathbb{R}^d$.
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Any of the following formulas for $\|x\|$ gives a norm:

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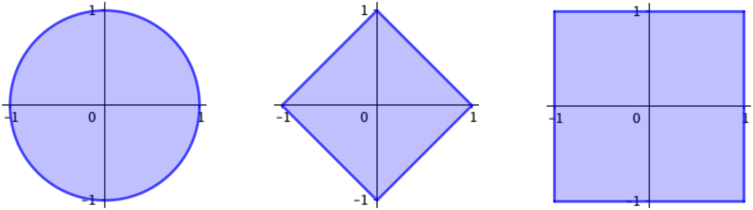
The unit ball $\{x \in \mathbb{R}^d ; \|x\| \leq 1\}$ is always a convex, symmetric set:



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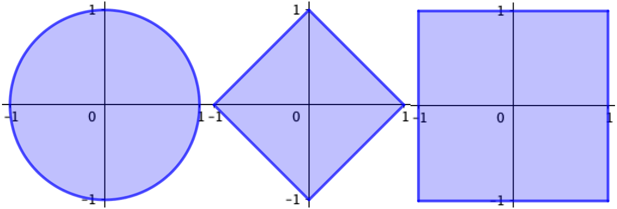

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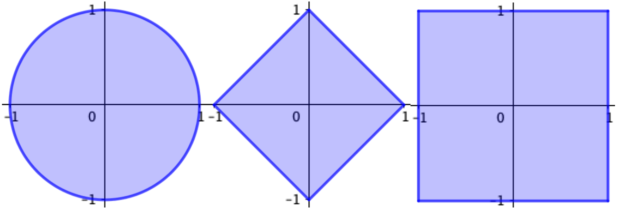

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So we can associate to each norm a closed, convex, symmetric, bounded set with 0 in its interior (let's call such sets *ball-like*).

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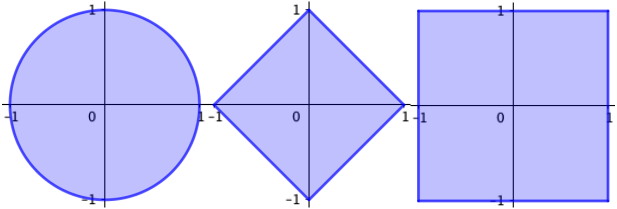

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Minkowski realized that we can also go in the other direction: every ball-like set is the unit ball of some norm, its so-called *Minkowski functional*.

Minkowski functional

Given a ball-like set $C \subseteq \mathbb{R}^d$, we can define a norm by:

$$\|x\|_C := \inf\{t \in [0,\infty); x \in Ct\}.$$

Moreover, C is the closed unit ball of $\|\cdot\|_C$.

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$$\|x\|_C := \inf \{t \in [0, \infty); x \in Ct\}.$$

Moreover, C is the closed unit ball of $\| \cdot \|_C$.
Thus, we have a bijective correspondence between norms and ball-like sets.

Convex sets and hyperplanes

Minkowski also studied the relationship between convex sets and hyperplanes.

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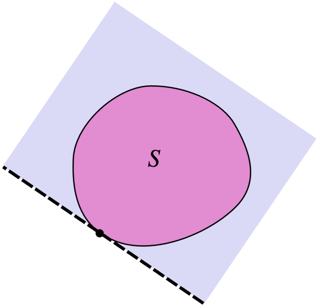
Supporting hyperplanes

Given a closed, convex set $C \subseteq \mathbb{R}^d$ and a point $p \in \partial C$, there is a hyperplane H passing through p and such that all the points of C are at one side of the hyperplane.



Hyperplane separation theorem

Given two closed, convex sets $A, B \subseteq \mathbb{R}^d$ whose interiors are disjoint, there is a hyperplane H such that A is at one side of H and B is at the other side.

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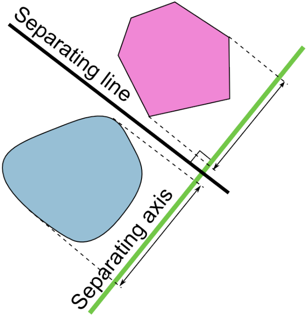

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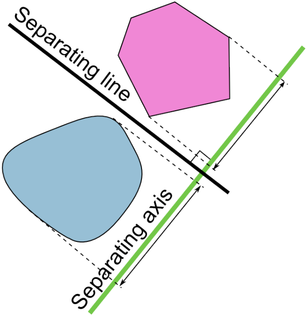

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The theorem also implies that every convex set is an intersection of hyperplanes.

Helly's theorem

The following result was discovered by Eduard Helly, and communicated to Radon, in 1913, and published by Radon (1921) and König (1922) with different proofs.

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Theorem (Helly, 1913)

Let $A_1, \ldots, A_n$ be subsets of $\mathbb{R}^d$, with $n \geq d+1$. If every $d+1$ of these sets have nonempty intersection, then $\bigcap_{i=1}^{n} A_i$ is nonempty.

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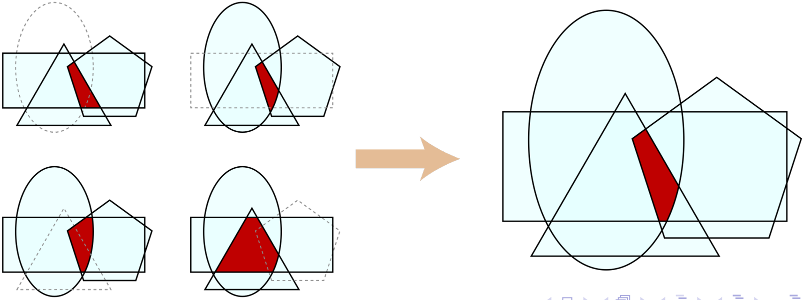
So for example in $\mathbb{R}^2$, if we have convex sets $X_1, \ldots, X_n$ and each three of them intersect, then all of them intersect.



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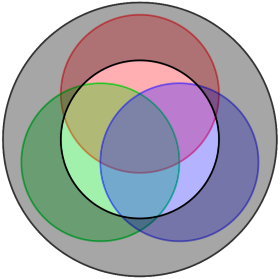

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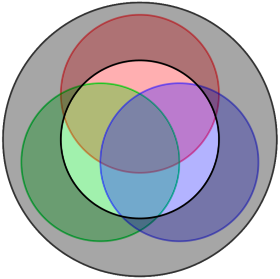
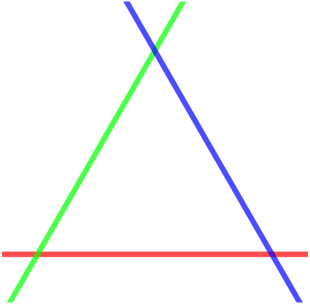


Or if intersections of each d sets is nonempty:



Proof of Helly's theorem

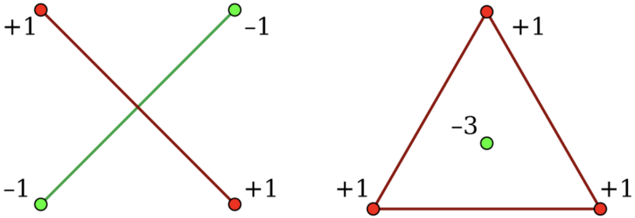
We follow Radon's proof of Helly's theorem. He used the following intermediate result:

Theorem (Radon)

Any points $x_1, \ldots, x_{d+2}$ in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect.

E.g. for four points in $\mathbb{R}^2$:

- If they form a convex quadrilateral, then the diagonals intersect.
- If not, one of them is inside the triangle formed by the other three.



Any points $x_1, \ldots, x_{d+2}$ in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect.

Proof. There are coefficients $a_1, \ldots, a_{d+2}$, not all zero, such that

$$\sum_{i=1}^{d+2} a_i x_i = 0 \quad \text{and} \quad \sum_i a_i = 0. \tag{1}$$

The a_i exist because (1) is a system of $d+1$ linear equations with $d+2$ variables.

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So there are $b_1, \ldots, b_{d+2} \geq 0$ ($b_i = |a_i|$) and a partition $\{1, \ldots, d+2\} = I \sqcup J$ such that

$$\sum_{i \in I} b_i x_i = \sum_{j \in J} b_j x_j \quad \text{and} \quad \sum_{i \in I} b_i = \sum_{j \in J} b_j$$

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After multiplying the coefficients b_i by a constant, we may assume $\sum_{i \in I} b_i = \sum_{j \in J} b_j = 1$, so the point $\sum_{i \in I} b_i x_i = \sum_{j \in J} b_j x_j$ is in the convex hulls of $(x_i)_{i \in I}$ and $(x_j)_{j \in J}$. $\square$

Proof of Helly's theorem

Let $A_1, \ldots, A_n$ be subsets of $\mathbb{R}^d$, with $n \geq d+1$. If every $d+1$ of these sets have nonempty intersection, then $\bigcap_{i=1}^{n} A_i$ is nonempty.

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