The Fundamental Theorem of Algebra: early history and first proof attempts.

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Reading Classics Seminar, 6 February 2024

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It is easy to see that the following are equivalent:

1. Every polynomial $p \in \mathbb{C}[z]$ of degree ≥ 1 has a root $z_0 \in \mathbb{C}$.
2. Every polynomial $p \in \mathbb{C}[z]$ of degree ≥ 1 can be expressed as a product of linear factors.
3. Every polynomial $p \in \mathbb{R}[z]$ of degree ≥ 1 can be expressed as a product of real polynomials of degrees 1 and 2.
4. Every polynomial $p \in \mathbb{R}[z]$ of degree ≥ 1 has a root $z_0 \in \mathbb{C}$.

The discovery of complex numbers

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For example, given the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d = 0$, here is a 'formula' for the roots of the polynomial. It is known as Cardano's formula.

$$\begin{aligned} x &= \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}. \end{aligned}$$

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So mathematicians started to accept that one could take square roots of negative numbers, and square/cubic roots of these numbers and so on. They coined the term 'imaginary numbers' for these expressions which were not real numbers but which could be used to compute anyways.

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$$\begin{aligned} x^4 + 1 &= \left(x + \frac{1+i}{\sqrt{2}}\right)\left(x + \frac{1-i}{\sqrt{2}}\right)\left(x + \frac{-1+i}{\sqrt{2}}\right)\left(x + \frac{-1-i}{\sqrt{2}}\right) \\ &= \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right). \end{aligned}$$

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It seems Leibniz' mistake was that when he tried to solve $x^4 + 1 = 0$, he obtained $x^4 = -1$, so $x^2 = \pm i$ so $x = \pm\sqrt{\pm i}$,but he didn't notice that $\sqrt{\pm i}$ could be written as $a + bi$, with $a, b \in \mathbb{R}$.

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It was also Euler who first gave a clear statement of the FTA, in a letter to Johann Bernoulli in 1739. In 1742, Euler wrote to Clairaut that the FTA is “indubitable, quoique je ne le puisse démontrer parfaitement” (undoubtable, even though I cannot prove it perfectly).

Why were mathematicians interested in the FTA?

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In the beginning of the 18^{th} century, mathematicians were trying to figure out how to integrate rational functions.
It was clear that if you can factor polynomials as products of factors of degree ≤ 2, then you can use partial fraction decomposition to integrate them.
This was the main motivation which led Euler and D'Alembert to tackle the FTA.

The first attempted proofs of the FTA

- D'Alembert, 1746.
- Euler, 1749.
- Foncenex, 1759.
- Lagrange, 1772.
- Laplace, 1795.
- Wood, 1798.
- Gauss, 1799 (he would try again in 1816, 1816 and 1849).
- Argand, 1806.

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Gauss' 1799 proof is sometimes said to be the first correct proof of the FTA, but some would argue that its gaps are more serious than the gaps in some previous proofs.

D'Alembert's proof attempt

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Basic idea: to prove that every real polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

has complex roots, he considers for each $r \in \mathbb{R}$ the polynomial

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For $r = 0$, we clearly have a root $x(0) = 0$.

D'Alembert argues that this root can be extended continuously to a complex root $x(r)$ for the polynomial p_r, thus proving that $p = p_{a_0}$ has a complex root.

Euler's proof attempts

Euler's proof attempt of the FTA was published in his article 'Recherches sur les racines imaginaires des équations', in 1749, but he had read the proof to the Berlin Academy of sciences in November 1746 (a month before D'Alembert's proof was sent to the Berlin Academy to be published).

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In pages 257-262, he tries an interesting approach. Firstly, he proves a few properties of complex numbers: the sum, difference, product, quotient and n^{th} roots of complex numbers are complex numbers. Then he argues as follows:

Euler's proof attempts

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Theoreme. XIV.

§. 76. *De quelque degré que ſoit une équation algébrique, toutes les racines imaginaires qu'elle peut avoir, ſont toujours compriſes dans cette forme générale* M+N$\sqrt{-1}$; *de ſorte que* M & N *ſont des quantités réelles.*

DEMONSTRATION.

Soit en general l'équation propoſée:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n\;3} + Dx^{n-4} + \&c. = 0.$$

& quoique nous ne ſoyons pas en état d'aſſigner la formule générale, qui en contient les racines, comme nous le ſommes pour les équations du ſecond, troiſiéme & quatriéme degré, il eſt pourtant certain, que cette formule ſera compoſée de pluſieurs ſignes radicaux, dont les quantités connuës A, B, C, D, E, &c. feront compliquées. On peut ...

Let in general be the proposed equation:
$x + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \cdots = 0.$
and although we are not in a position to assign the general formula which contains the roots, as we are for the equations of the second, third and fourth degree, it is nevertheless certain that this formula will be composed of several radical signs, including the known quantities A, B, C, D, E,...

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This would prove the FTA for all polynomials of degree 2^n, and thus for all polynomials.

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It seems the only problem with Laplace's proof is also that he assumes the existence of 'imaginary' roots.

Gauss' 1799 proof attempt

In his PhD thesis in 1799, Gauss first pointed out some of the problems he found in the proofs of D'Alembert, Euler, Foncenex and Laplace. Then he proceeded to give his own proof.

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In order to prove the FTA, Gauss considered a polynomial with real coefficients: $p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$.
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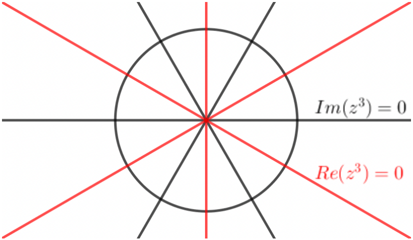

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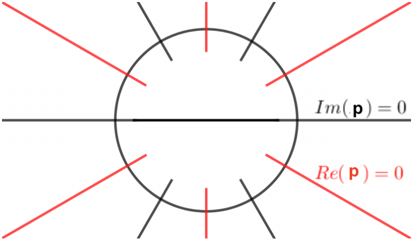

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It seems to be well demonstrated that an algebraic curve neither ends abruptly (as it happens in the transcendental curve $y = 1/\log x$*), nor lose itself after an infinite number of windings in a point (like a logarithmic spiral). As far as I know nobody has ever doubted this, but if anybody requires it, I take it on me to present, on another occasion, an indubitable proof . . .*

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Using this and the way the intersections of the curves with C_r are interleaved, he completed the argument (he seems to use some version of the Jordan curve theorem?).

Argand

In 1806, in his article *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, Argand introduces the 'Argand diagram', which is the standard way in which we represent complex numbers, using real numbers as the x-axis and imaginary numbers as the y-axis.

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Argand states the FTA in its modern form: every complex polynomial can be expressed as a product of linear factors. To prove the FTA, Argand argued as follows.

Argand's proof attempt

Suppose that $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ $(n \geq 1)$ is the polynomial for which we want to find a root.

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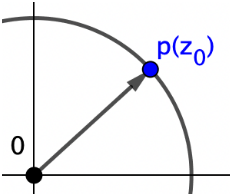

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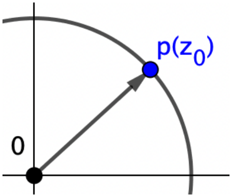

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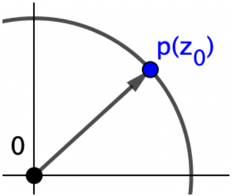

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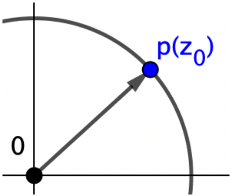

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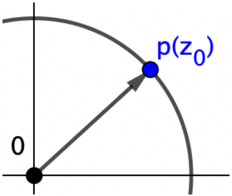

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Note that when h is very small, the terms $A_2 h^2 + \cdots + h^n$ are very small compared to $A_1 h$ (unless $A_1 = 0$, but that case can be done similarly).
More specifically, $|A_1 h| > |A_2 h^2 + \cdots + h^n|$ for any $h \in \mathbb{C}$ close enough to 0.

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Now the strategy is simple: choose h to be a small complex number such that $A_1 h$ has direction opposite to $\overrightarrow{0p(z_0)}$.

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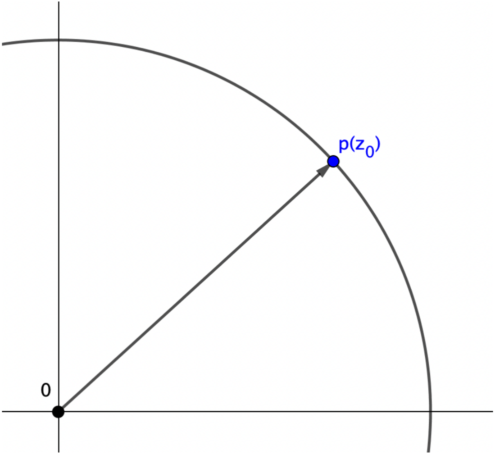

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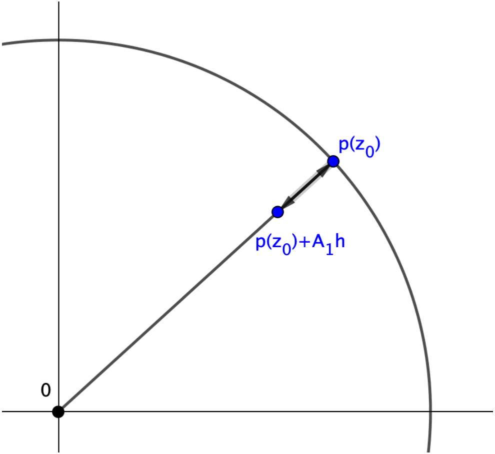

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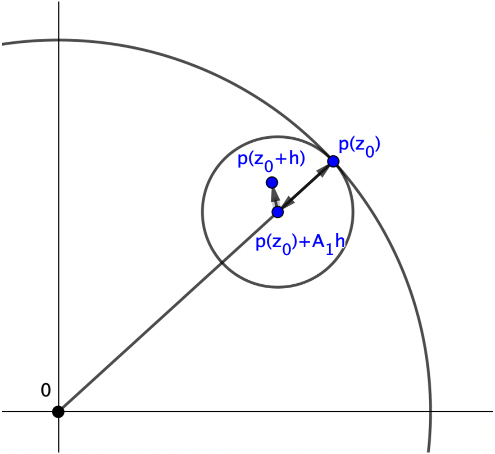

This forces $|p(z_0 + h)|$ to be smaller than $|p(z_0)|$, as we wanted.

Bibliography

1. Mac Tutor - The fundamental theorem of algebra, Jean Robert Argand, Caspar Wessel.
2. Christopher Baltus. *D'Alembert's proof of the fundamental theorem of algebra.* Historia Mathematica 31 (2004) 414–428.
3. William Dunham. *Euler and the Fundamental Theorem of Algebra.*
4. Jean Robert Argand. *Reflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théoréme d'analise.* Translated by Michael Bertrand. arXiv:2212.01283v1 [math.HO] 9 Oct 2022.
5. Soham Basu and Daniel J. Velleman. *On Gauss's First Proof of the Fundamental Theorem of Algebra.* arXiv:1704.06585v1 [math.CV] 21 Apr 2017.
6. Harel Cain. *C. F. Gauss' proofs of the fundamental theorem of Algebra.*

Proof attempts

- D'Alembert, J., 1746. Recherches sur le calcul intégral. Histoire de l'Acad. Royale Berlin (1748), 182–224.
- Euler, L., 1749. Recherches sur les racines imaginaires des équations. Mem. Berlin (1751). In: Opera Series 1, vol. 6, 78–150.
- Foncenex, D., 1759. Réflexions sur les quantités imaginaires. Miscellanea Taurinensis 1, 113–146.
- Lagrange, J.L., 1772. Sur la forme des racines imaginaire des équations. N. Mem. Berlin (1774). In: Oeuvres de Lagrange 3, 479–516.
- Laplace, P.-S., 1812. Lecons de mathématiques donnée à l'École Normale en 1795. Journal de l'École Polytechnique, VIIe et VIIIe Cahiers. Reprinted in: Oeuvres Compléte de Laplace. Gauthier–Villars, Paris, 1912, 14, 347–377.
- C. F. Gauss. *Demonstratio nova theorematis omnem functionem algebraicam ratio- nalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse.* PhD thesis, Universität Helmstedt, 1799. In Werke III, 1–30.
- R. Argand, 1806. Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques. (Privately distributed between 1806-1813)