Reading Classics Seminar, 7 November 2023

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This Essay was written in 1737 and published in 1744 in the *Proceedings of the National Academy of St. Petersburg*.
A highlight of the article is that it contains the first proof that e is irrational.

Continued fractions

Continued fractions are expressions of the following form:

$$a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \text{etc}}}}} \tag{1}$$

They can be represented by two sequences of positive numbers: the 'numerators' $\alpha, \beta, \gamma, \ldots$ and the 'denominators' $a, b, c, \ldots$.
Example: Brouncker asserted that

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{etc}}}}}} \tag{2}$$

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Similarly, $a + \frac{\alpha}{b}$ is an upper bound for s,$a + \frac{\alpha}{b + \frac{\beta}{c}}$ is a lower bound for s, etc.

Thus continued fractions are useful for approximating numbers, as they give a sequence of upper and lower bounds:

$$a \quad a+\frac{\alpha}{b} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} \quad \text{etc.}$$

These approximations can be expressed as simple fractions:

$$\frac{a}{1} \quad \frac{ab+\alpha}{b} \quad \frac{abc+\alpha c+\beta a}{bc+\beta} \quad \frac{abcd+acd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} \quad \text{etc}$$

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Euler gives a rule to compute the approximations:

$$\begin{array}{cccccc}
a & b & c & d & e & \\
\frac{1}{0} & \frac{a}{1} & \frac{ab+\alpha}{b} & \frac{abc+\alpha c+\beta a}{bc+\beta} & \frac{abcd+acd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc} \\
\alpha & \beta & \gamma & \delta & \varepsilon &
\end{array}$$

Convergence of continued fractions

By computing the differences between the fractions above, Euler obtains an expression for the continued fraction as an alternating series:

$$
\begin{aligned}
a &+ \frac{\alpha}{1 \cdot b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} \\
&- \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+bc\delta+\beta\delta)} + \text{etc}
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Some of Euler's observations:
'This series converges rapidly and is quite suitable for the approximate computation of its value.'
'Therefore it is clear that a continued fraction converges fastest when its numerators $\alpha, \beta, \gamma, \dots$ are small and its denominators $a, b, c, \dots$ are large.'

Simple continued fractions

Continued fractions of the form

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc}}}}}, \tag{3}$$

where $a, b, c, \dots \in \mathbb{Z}^{>0}$, can be expressed as

$$\begin{aligned} a + \frac{1}{b} - \frac{1}{b(bc+1)} + \frac{1}{(bc+1)(bcd+d+b)} \\ - \frac{1}{(bcd+d+b)(bcde+de+be+bc+1)} + \text{etc}, \end{aligned}$$

so they are convergent.

Simple continued fractions

Some comments from Euler:

- About transforming an ordinary fraction $\frac{a}{b}$ into a simple continued fraction:
  'To find such a continued fraction, it suffices to assign the denominators, since we set all numerators equal to one. In fact, these will be derived from the numerator and denominator of the given fraction by executing the customary operation for finding their greatest common divisor.'

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- Continued fractions of rational and irrational numbers:
  'Every finite fraction whose numerators and denominators are finite whole numbers may be transformed into a continued fraction of this kind which is truncated at a finite level. On the other hand, a fraction whose numerator and denominator are infinitely large numbers (which are given for irrational and transcendental quantities) will go across to a continued fraction running to infinity.'

Simple continued fractions

- About best rational approximations to a number $\frac{A}{B}$:
  'Moreover, if any of these fractions will be examined, no other fraction which lies closer to the value $\frac{A}{B}$ can be expressed with smaller numbers. In this way the following problem is conveniently solved: "To convert a fraction composed of large numbers into a simpler one which approximates it more closely than can be done with numbers which are no larger."'

The Gregorian calendar

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The ratio of 24h to 5h, 49', 8" is $\frac{21600}{5237}$. Thus, in every interval of 21600 years there should be 5237 leap years.

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This is the expression of $\frac{21600}{5237}$ as a simple continued fraction:

$$\frac{21600}{5237} = 4 + \cfrac{1}{8 + \cfrac{1}{31 + \cfrac{1}{21}}}.$$

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Which gives the following principal convergents

$$\begin{array}{ccccc}
4 & 8 & 31 & 21 & \\[6pt]
\frac{1}{0} & \frac{4}{1} & \frac{33}{8} & \frac{1027}{249} & \frac{21600}{5237} \\[6pt]
1 & 1 & 1 & &
\end{array}$$

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Numbers of the form $\frac{33k+4}{8k+1} = 4 + \cfrac{1}{8 + \cfrac{1}{k}}$, for $k = 1, \ldots, 31$.

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- Multiples of 4 are leap years, except...
- Multiples of 100 are not leap years, except...
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Error in the Julian Calendar: 1 day every $\sim$ 128 years.
Error in the Gregorian Calendar: 1 day every $>$ 3000 years.
'Most accurately, however, the calendar will be reconciled with the sun if in the interval of 21,600 years one year which ought to be a leap year according to the Gregorian calendar is changed to an ordinary one.'

Periodic continued fractions

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Then $x - a = \cfrac{1}{b + \cfrac{1}{b + \text{etc}}}$, so $x - a = \frac{1}{b+(x-a)}$, so

$$x = a - \frac{b}{2} + \sqrt{1 + \frac{b^2}{4}}.$$

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$$x = a - \frac{b}{2} + \sqrt{1 + \frac{b^2}{4}}.$$

Setting $b = 2a$, we get $\sqrt{1 + a^2} = a + \cfrac{1}{2a + \cfrac{1}{2a + \cfrac{1}{2a + \text{etc}}}}$.

Periodic continued fractions

Similarly, for fractions of the form

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{c + \text{etc}}}}}$$

and setting $c = 2a$, we get the solution $x = \sqrt{a^2 + \frac{2a}{b}}$.
In general, the values of periodic continued fractions are solutions of quadratic equations.

The number e

Using the approximation $e \approx 2.71828182845904$, Euler obtained the start of the simple continued fraction of e:

$$e = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\text{etc}}}}}}}}}}$$

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Its denominators apparently form an arithmetic progression, $2, 4, 6, \ldots$, interrupted by ones. Euler proves that this is actually the case, thus proving for the first time that e is an irrational number (!).

Arithmetic progressions

Similar interrupted/uninterrupted arithmetic progressions occur in the simple continued fraction expressions for $\sqrt{e}$, $\frac{\sqrt[3]{e}-1}{2}$, $\frac{e^2-1}{2}$ and $\frac{e+1}{e-1}$.

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Euler finds a relation between continued fractions with interrupted and uninterrupted arithmetic progressions: given numbers $m, n, a, b, c, d, \ldots,$ we have

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\text{etc}}}}}}} = \frac{1}{mn+1}\left((mn+1)a+n+\cfrac{1}{(mn+1)b+m+n+\cfrac{1}{(mn+1)c+m+n+\text{etc}}}\right)$$

The number e is irrational

In sections 31-32 of his essay, Euler proves that the continued fraction $a + \cfrac{1}{3a + \cfrac{1}{5a + \cfrac{1}{7a + \text{etc}}}}$, has the value $\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1}$.

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$$\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1} = 2s + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + \text{etc}}}}.$$

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Using the translation from interrupted to uninterrupted arithmetic progressions, we find that

The number e is irrational

$$\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1}=2s-1+\cfrac{2}{1+\cfrac{1}{1+\cfrac{1}{3s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5s-1+\text{etc}}}}}}}$$

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And now noting that $e^{\frac{1}{s}} = 1+\frac{2}{\left(\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1}\right)-1}$, we obtain

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The number e is irrational

Thus for $s = 1$ we get

$$e = 1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \text{etc}}}}}}}} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \text{etc}}}}}},$$

and we conclude that e is irrational.

Sketch of the rest of the proof

So, how did Euler find that $a + \cfrac{1}{3a + \cfrac{1}{5a + \cfrac{1}{7a + \text{etc}}}} = \dfrac{e^{\frac{2}{a}} + 1}{e^{\frac{2}{a}} - 1}$?

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More generally, he let $s = s(a, n)$ denote the following continued fraction

$$s(a,n) = a + \cfrac{1}{(1+a)n + \cfrac{1}{(1+2n)a + \cfrac{1}{(1+3n)a + \cfrac{1}{(1+4n)a + \text{etc}}}}}$$

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He checks that then we have

$$s=\frac{a+\frac{1}{na}+\frac{1}{1\cdot2\cdot1(1+n)n^2a^3}+\frac{1}{1\cdot2\cdot3\cdot1(1+n)(1+2n)n^3a^5}+\text{etc}}{1+\frac{1}{1(1+n)na^2}+\frac{1}{1\cdot2(1+n)(1+2n)n^2a^4}+\frac{1}{1\cdot2\cdot3(1+n)(1+2n)(1+3n)n^3a^6}+\text{etc}}.$$

Sketch of the rest of the proof

Letting $z = \frac{1}{a\sqrt{n}}$, this can be rewritten as

$$s(z,n) = \frac{1}{z\sqrt{n}} \cdot \frac{1 + \frac{z}{1\cdot 1} + \frac{z^2}{1\cdot 2\cdot 1(1+n)} + \frac{z^3}{1\cdot 2\cdot 3\cdot 1(1+n)(1+2n)} + \text{etc}}{1 + \frac{z}{1(1+n)} + \frac{z^2}{1\cdot 2(1+n)(1+2n)} + \frac{z^3}{1\cdot 2\cdot 3(1+n)(1+2n)(1+3n)} + \text{etc}}.$$

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From the power series expressions of t, u one can obtain that $\frac{dt}{dz} = u$ and $\frac{du}{dz} = \frac{t-u}{nz}$.

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$$= \frac{1}{z\sqrt{n}} \cdot \frac{t(n,z)}{u(n,z)}.$$

From the power series expressions of t, u one can obtain that $\frac{dt}{dz} = u$ and $\frac{du}{dz} = \frac{t-u}{nz}$.
Using that and letting $v = \frac{t}{u}$, one can deduce that $\frac{dv}{dz} = 1 + \frac{v-v^2}{nz}$.

Sketch of the rest of the proof

Letting $z = \frac{1}{a\sqrt{n}}$, this can be rewritten as

$$s(z,n) = \frac{1}{z\sqrt{n}} \cdot \frac{1 + \frac{z}{1 \cdot 1} + \frac{z^2}{1 \cdot 2 \cdot 1(1+n)} + \frac{z^3}{1 \cdot 2 \cdot 3 \cdot 1(1+n)(1+2n)} + \text{etc}}{1 + \frac{z}{1(1+n)} + \frac{z^2}{1 \cdot 2(1+n)(1+2n)} + \frac{z^3}{1 \cdot 2 \cdot 3(1+n)(1+2n)(1+3n)} + \text{etc}}.$$

$$= \frac{1}{z\sqrt{n}} \cdot \frac{t(n,z)}{u(n,z)}.$$

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We can then change to two new variables $q = \frac{v}{z^{1/n}}$ and $r = z^{1/n}$, and the equation becomes the Ricatti equation $\frac{dq}{dr} = nr^{n-2} - q^2$.

Sketch of the rest of the proof

For our purposes we are interested in the case $n = 2$, so that the equation is $\frac{dq}{dr} = 2 - q^2$, with solution $q = \frac{(e^{2r\sqrt{2}}+1)\sqrt{2}}{e^{2r\sqrt{2}-1}}$.

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We can now use that $n = 2$ to solve for r, obtaining $r = \frac{1}{a\sqrt{2}}$.And then, undoing the changes of variables, we can compute the value of the original continued fraction s:

$$\frac{e^{2/a}+1}{e^{2/a}-1} = arq = \left(\frac{1}{z\sqrt{n}}\right)\left(z^{1/n}\right)\left(\frac{v}{z^{1/n}}\right) = \frac{1}{z\sqrt{n}}v = \frac{1}{z\sqrt{n}} \cdot \frac{t}{u} = s.$$